AN

INTRODUCTION

TO

MENSURATION, &c.

[PRICE THREE SHILLINGS AND SIXPENCE.]

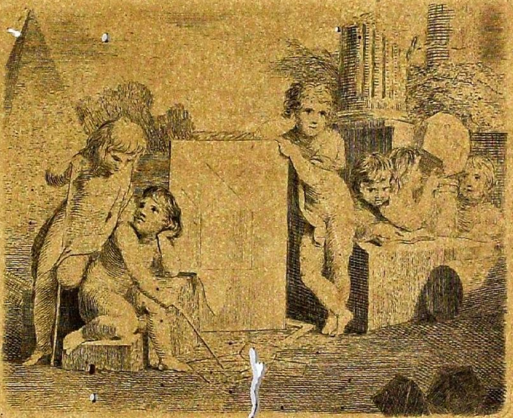
AN INTRODUCTION TO MENSURATION, AND PRACTICAL GEOMETRY,

WITH NOTES, CONTAINING THE REASON OF EVERY RULE.

By JOHN BONNYCASTLE,

OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

The FOURTH EDITION, corrected and improved.



LONDON: Printed for J. JOHNSON, Nº 72, ST. PAUL'S CHURCH YARD. 1794.

PREFACE.

THE ART OF MEASURING, like all other useful inventions, appears to have been the offspring of want and necessity; and to have had its origin in those remote ages of antiquity, which are far beyond the reach of credible and authentic history. Egypt, the fruitful mother of almost all the liberal sciences, is imagined likewise to have given birth to GEOMETRY or MENSURATION; it being to the inundations of the Nile that we are said to be indebted for this most perfect and delightful branch of human knowledge.

After the overflowings of the river had deluged the country, and all artificial boundaries and land-marks were destroyed, there could have been no other method of ascertaining individual property, than by a previous knowledge of its figure and dimensions. From this circumstance, it appears highly probable, that Geometry was first known and cultivated by the ancient Egyptians; as being the only science which could administer to their wants, and furnish them with the assistance they required. The name itself signifies properly the *art of measuring the earth*; which serves still further to confirm this opinion; especially as it is well known that many of the ancient mathematicians applied their geometrical knowledge entirely to that purpose; and that even the Elements of Euclid, as they now stand, are only the theory from whence we obtain the rules and precepts of our present more mechanical practice.

But to trace the sciences to their first rude beginnings, is a matter only of learned curiosity, which

could afford but little gratification to readers in general. It is of much more consequence to the rising generation to be informed that, in their present improved state, they are exceedingly useful and important. And in this respect, the art I have undertaken to elucidate is inferior to none, arithmetic only excepted. Its use in most of the different branches of the Mathematics is so general and extensive, that it may be justly considered as the mother and mistress of all the rest, and the source from whence were derived the various properties and principles to which they owe their existence.

As a testimony of this superior excellence, I need only mention a few of those who have studied and improved it; in which illustrious catalogue we have the names of Euclid, Archimedes, Thales, Anaxagoras, Pythagoras, Plato, Apollonius, Philo and Ptolemy, amongst the ancients; and Huygens, Wallis, Gregory, Halley, the Bernoullies, Euler, Liebnitz, and Newton, among the moderns; all of whom applied themselves to particular parts of it, and greatly enlarged and improved the subject. To the latter especially we are indebted for many valuable discoveries in the higher branches of the art; which have not only enhanced its dignity and importance, but rendered the practical application of it more general and extensive.

The degree of estimation in which the art was held by these and other eminent characters, will, in general, it is apprehended, be thought a sufficient encomium on its merits. But, for the sake of young people, and those of a confined education, it may not be amiss to give a few more instances of its advantage, and shew that its importance in trade and business is not inferior to its dignity as a science. Artificers of almost all denominations are indebted to this invention for the establishment of their several occupations, and the perfection and value of their workmanship. Without its assistance all the great and noble works

of Art would have been imperfect and useless. By this means the architect lays down his plan, and erects his edifice; bridges are built over large rivers; ships are constructed; and property of all kinds is accurately measured, and justly estimated. In short, most of the elegancies and conveniencies of life owe their existence to this art, and will be multiplied in proportion as it is well understood, and properly practised.

From this view of the subject, it is hardly to be accounted for, that, in a commercial nation, like our own, an art of such general application should have been so greatly neglected. Mechanics of all kinds, it is well known, are but ill acquainted with its principles; and those who have been the best qualified to afford them any assistance, have thought it beneath their attention. Till within a few years past there could not be found a regular treatise upon this subject in the English language. Some particular branches, it is true, had been greatly cultivated and improved; but these were only to be found in their miscellaneous state, interspersed through a number of large volumes, in the possession of but few, and in a form and language totally unintelligible to those for whom they were more immediately necessary.

Dr. Hutton was the first person, in this country, who undertook to collect these scattered fragments, and to treat of the subject in a scientific, methodical manner. A small treatise by Hawney, and some others of little note, had indeed been long in the hands of the public; but these were extremely defective, both in matter and method; neither the principles nor practice of the art being properly or clearly explained. Before the publication of the treatise abovementioned, Mr. Robertson's may be considered as the only book, of any value, that could be consulted, either by the artizan or mathematician; and had he given the theory as well as the practice of the art,

art, and divested his rules and examples of their algebraical form, there would have been no want of any other elementary treatise.

To these two writers I am greatly indebted for many things in the following pages, and am ready to acknowledge, that I have used an unreserved freedom in selecting from their works, wherever I found them to answer my purpose. To Dr. Hutton I am particularly obliged, and am so far from desiring to supercede the use of his performance by this publication, that I only wish it to be thought a useful introduction to it. His treatise is excellent in its kind; and had it been as well calculated for the use of the uninformed Artist as it is for the Mathematician, the following compendium had certainly never been published.

The method I have observed, in composing this work, is that which was used in the "*Scholar's Guide to Arithmetic*;" and, as my object has been to facilitate the acquirement of the same kind of useful knowledge, I am not without hopes of its being received with equal candour and approbation.

In school-books, and those designed for the use of learners, it has always appeared to me, that plain and concise rules, with proper exercises, are entirely sufficient for the purpose. In science, as well as in morals, example will ever inforce and illustrate precept; for this reason an operation, wrought out at length, will be found of more service to beginners than all the tedious directions and observations that can possibly be given them. From constant experience I have been confirmed in this idea; and it is in pursuance of it that I have formed the plan of this publication. I have not been ambitious of adding much new matter to the subject; but only to arrange and methodize it in a manner more easy and rational than had been done before.

The text part of the work contains the rules in words at length, with examples to exercise them;

and, in order that the learner may not be perplexed and interrupted in his progrefs, the remarks and demonftrations are confined to the notes, and may be confulted or not, as fhall be thought neceffary. To thofe who would wifh not to take things upon truft, but to be acquainted with the grounds and *rationale* of the operations they perform, they will be found extremely ferviceable; and for this purpofe I have endeavoured to make them as eafy as the nature of the fubject would admit. But they can be confulted only by fuch as have made a previous acquaintance with feveral other branches of mathematical learning.

Some of the moft difficult rules relating to the furfaces of folids, &c. could not be conveniently given, but by means of algebraical theorems; and as this was foreign to my purpofe, I have not fcrupled to omit them; being well perfuaded that what is done upon that head will be fully fufficient to anfwer moft practical purpofes. In the Practical Geometry likewife, which is prefixed to this treatife, fuch problems only are introduced as were known to be moft intimately connected with the fubject. And as this part of the work is a proper and neceffary introduction to the reft, I have fpared no pains in making it as clear and intelligible as poffible.

Upon the whole, I have endeavoured to confult the wants of the learner, more than thofe of the man of fcience. And if I have fucceeded in this refpect, my purpofe is anfwered. I have not fought for reputation as a mathematician, but only to be ufeful as a tutor.

N. B. The favourable reception this Work has met with, has induced me in this Edition to make fuch Alterations and Additions as have fince occurred to me, and which are fuch as I hope will render it ftill more acceptable to the Public.

Royal Academy, Woolwich,
March 1, 1794.

CON-

CONTENTS.

	Page
PRACTICAL GEOMETRY.	
Definitions, - - - -	1
Geometrical problems, - -	9
OF THE MENSURATION OF SUPERFICIES, -	41
To find the area of a parallelogram, -	ibid.
To find the area of a triangle, - -	45
To find the fides of a right-angled triangle, -	48
To find the area of a trapezium, - -	50
To find the area of a regular polygon, -	53
Of a circle, and its parts, - -	57
To find the area of a circular ring, -	80
To find the areas of lunes, - -	82
To find the area of an irregular polygon, -	83
Of the conic fections, - -	86
Definitions, - - - -	ibid.
To defcribe an ellipfis, - - -	90
Of an ellipfis, and its parts, - -	91
To defcribe a parabola, - -	99
Of a parabola, and its parts, - -	101
To conftruct an hyperbola, - -	106
Of an hyperbola, and its parts, - -	107
OF THE MENSURATION OF SOLIDS.	
Definitions, - - - -	117
To find the folidity of a cube, - -	120
To find the folidity of a parallelopipedon, -	121
To find the folidity of a prifm, - -	122
Of a cylinder, and its parts, - -	124
Of the cone and pyramid, and their parts, -	128
To find the folidity of a wedge, - -	139

To

CONTENTS.

	Page
To find the ſolidity of a priſmoid, - -	141
Of a ſphere, and its parts, - -	142
Of a circular ſpindle, and its parts, - -	150
Of a ſpheroid, and its parts, - -	156
Of an elliptic ſpindle, and its parts, -	165
Of a parabolic conoid, and its parts, -	170
Of a parabolic ſpindle, and its parts, -	173
Of an hyperboloid, and its parts, - -	176
Of the regular bodies, - - -	181
Of cylindric rings, - - -	190
Of the weight of balls and ſhells, &c. -	193
Of the piling of ſhot, - - -	208
Of Artificers work, - - -	214
Of vaults and domes, &c. - - -	233
Of the Carpenters rule, - -	243
Of board and timber meaſure, - -	247
Of ſpecific gravity, - - -	259
Miſcellaneous queſtions, - -	265
A table of the ſegments of a circle, - -	271

TABLES

TABLES

OF THE

DIFFERENT MEASURES USED IN THIS WORK.

Lineal Measures.		*Square Measures.*	
12 inches make	1 foot.	144 inches make	1 foot.
3 feet - - -	1 yard.	9 feet - - -	1 yard.
6 feet - - -	1 fathom.	36 feet - - -	1 fathom.
$16\frac{1}{2}$ feet, or $5\frac{1}{2}$ yards, - -	1 pole, or rod.	$272\frac{1}{4}$ feet, or $30\frac{1}{4}$ yds. - -	1 pole, or rod.
40 poles - -	1 furlong.	1600 poles - - -	1 furlong.
8 furlongs - -	1 mile.	64 furlongs - -	1 mile.

Note, The chain made uſe of in meaſuring land, commonly called Gunter's chain, is 4 poles, or 22 yards in length, and conſiſts of 100 equal links, each link being $\frac{22}{100}$ of a yard, or 7.92 inches long.

An acre of land is alſo equal to 10 ſquare chains; that is, 10 chains in length, and 1 in breadth; or it is 4840 ſquare yards, or 160 ſquare poles, or 100,000 ſquare links.

Note alſo, that in Land Meaſure,		And in Cubic Meaſure,	
40 perches, or ſquare poles make	1 rood.	1728 inches make	1 foot.
4 roods - - - - - - -	1 acre.	27 feet - - -	1 yard.
		$166\frac{3}{4}$ yards - - -	1 pole.

PRAC-

PRACTICAL GEOMETRY.

DEFINITIONS.

1. *GEOMETRY* is that ſcience which treats of the deſcriptions and properties of magnitudes in general.

2. A *point* is that which has no parts, or dimenſions.

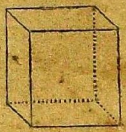
3. A *line* is length without breadth; and its bounds or extremes are points.

4. A *right line* is that which lies evenly between its extreme points.

5. A *ſuperficies* is that which has length and breadth only; and its bounds or extremes are lines.

6. A *plane ſuperficies* is that which touches, in every part, any right line that can be drawn in that ſuperficies.

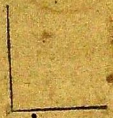
7. A *solid* is that which has length, breadth and thickness; and its bounds, or extremes, are superficies.



8. A *plane rectilineal angle* is the inclination or opening of two right lines which meet in a point.

9. One line is said to be *perpendicular* to another, when it makes the angles on both sides of it equal to each other.

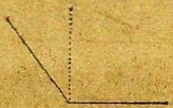
10. A *right angle* is that which is formed by two lines that are perpendicular to each other.*



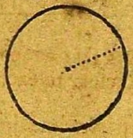
* Any angle differing from a right one, is, by some writers, called an *oblique angle*.

11. An *acute angle* is that which is leſs than a right angle.

12. An *obtuſe angle* is that which is greater than a right angle.



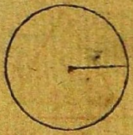
13. A *circle* is a plane figure, formed by the revolution of a right line, about one of its extremities, which remains fixed.*



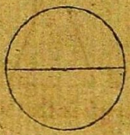
14. The centre of a circle, is the point about which it is deſcribed; and the circumference is the line or boundary by which it is contained.

* N. B. The circumference itſelf, for the ſake of conciſeneſs, is ſometimes called a circle.

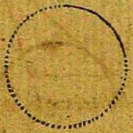
15. The *radius* of a circle is a right line drawn from the centre to the circumference.



16. The *diameter* of a circle is a right line paſſing through the centre, and terminated both ways by the circumference.



17. An *arc* of a circle is any part of its periphery or circumference.



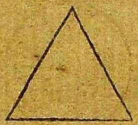
18. A *chord* is a right line joining the extremities of an arc.

N.B. A ſemicircle is half the circle, and a quadrant the quarter of it.

19. All

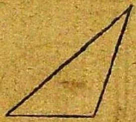
19. All plane figures bounded by three right lines are called *triangles*.

20. An *equilateral triangle* is that whose three sides are all equal.



21. An *isosceles triangle* is that which has only two of its sides equal.

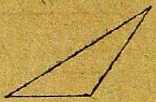
22. A *scalene triangle* is that which has all its three sides unequal.



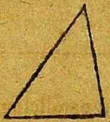
23. A *right-angled triangle* is that which has one right angle.*

* Any triangle differing from a right angled one, is called *an oblique angled triangle*.

24. An *obtuse-angled triangle* is that which has one obtuse angle.



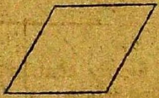
25. An *acute-angled triangle* is that which has all its angles acute.



26. All plane figures, bounded by four right lines, are called *quadrangles*, or *quadrilaterals*.

27. A *square* is a quadrilateral, whose sides are all equal, and its angles all right angles.

28. A *rhombus* is a quadrilateral whose sides are all equal, but its angles not right angles.*



* This figure, by working mechanics, is sometimes called a lozenge.

29. A

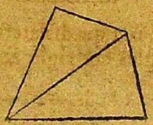
29. A *parallelogram* is a quadrilateral whoſe oppoſite ſides are parallel.

30. A *rectangle* is a parallelogram whoſe angles are all right angles.

31. A *rhomboid* is a parallelogram whoſe angles are not right angles.

32. All other four-ſided figures, beſides theſe, are called *trapeziums*.

33. A right line joining any two oppoſite angles of a four-ſided figure is called the *diagonal*.



34. All plane figures contained under more than four ſides are called *polygons*.

35. Polygons having five ſides, are called *pentagons*; thoſe of ſix ſides, *hexagons*; thoſe of ſeven, *heptagons*; and ſo on.

 36. A

36. A *regular polygon* is that whose angles, as well as sides, are all equal.

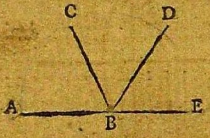
37. *Parallel right lines* are such as are every where at an equal distance; or which if infinitely produced would never meet.

---

38. The *base* of any figure is that side on which it is supposed to stand; and the *altitude* is the perpendicular falling upon it from the opposite angle.

39. In a right-angled triangle the side opposite to the right angle is called the *hypothenuse*; and the other two sides are called *legs*.

40. An angle is usually denoted by three letters, the one which stands at the angular point being always to be read in the middle.



* 41. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees*; each degree into 60 equal parts, called *minutes*; and so on.

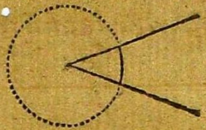
42. The *measure* of any right-lined angle is an arc of a circle contained between the two lines

---

* This and the following definition are used only in Practical Geometry.

which

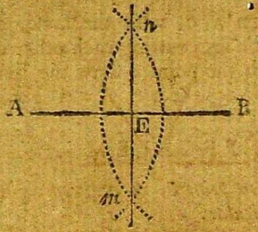
which form that angle, the angular point being the centre,



Note. The angle is estimated by the number of degrees contained in the arc; whence a right angle is an angle of 90 degrees, or $\frac{1}{4}$ of the circumference.

PROBLEM I.*

To divide a given line AB into two equal parts.



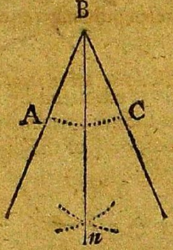
1. From the points A and B, as centres, with any distance greater than half A B, describes arcs cutting each other in *n* and *m*.

2. Through these points draw the line *n* E *m*, and the point E, where it cuts A B, will be the middle of the line required.

* The demonstrations of most of these problems may be found in Euclid's Elements.

PROBLEM II.

To divide a given angle A B C into two equal parts.



1. From the point B, with any radius, deſcribe the arc A C.

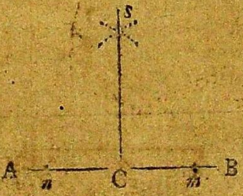
2. And from A, C, with the ſame, or any other radius, deſcribe arcs cutting each other in *n*.

3. Through the point *n* draw the line B *n*, and it will biſect the angle A B C, as was required.

PROBLEM III.

From a given point C, in a given right line A B, to erect a perpendicular.

CASE I. *When the point is near the middle of the line.*



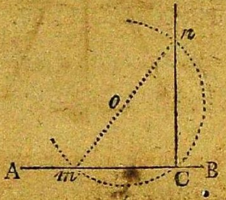
1. On

1. On each ſide of the point c take any two equal diſtances c *n*, c *m*.

2. From *n* and *m*, with any radius greater than c *n* or c *m*, deſcribe arcs cutting each other in *s*.

3. Through the point *s*, draw the line *s* c, and it will be the perpendicular required.

Case II. *When the point is at, or near, the end of the line.*

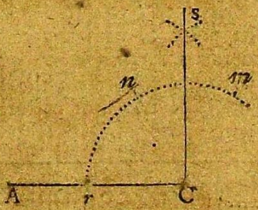


1. Take any point *o*, and with the radius or diſtance *o* c, deſcribe the arc *m* c *n*, cutting A B in *m* and c.

2. Through the centre *o*, and the point *m*, draw the line *m o n*, cutting the arc *m* c *n* in *n*.

3. From the point *n*, draw the line *n* c, and it will be the perpendicular required.

Another method.



1. From the point C, with any radius, deſcribe the arc *r n m*, cutting the line A C in *r*.

2. With the ſame radius, and *r* as a centre, croſs the arc in *n*; and from *n*, in like manner, croſs it in *m*.

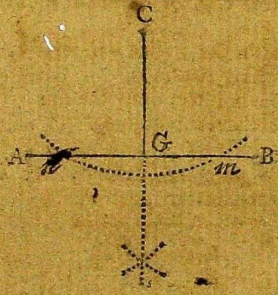
3. From the points *n* and *m*, with the ſame, or any other radius, deſcribe arcs cutting each other in *s*.

4. Through the point *s*, draw the line *s* C, and it will be the perpendicular required.*

PROBLEM IV.

From a given point C, out of a given line AB, to let fall a perpendicular.

CASE I. *When the point is nearly oppoſite to the middle of the line.*



1. From the point C, with any radius, deſcribe the arc *n m*, cutting A B in *n* and *m*.

2. From the points *n*, *m*, with the ſame, or any other radius, deſcribe two arcs cutting each other in *s*.

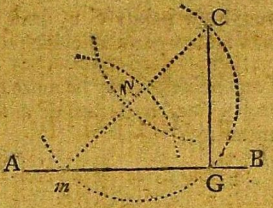
---

* Another method of raiſing a perpendicular from any point in a given line may be ſeen at page 40.

3. Through

3. Through the points C, *s*, draw the line C G *s*, and C G will be the perpendicular required.

CASE II. *When the point is nearly opposite to the end of the line.*



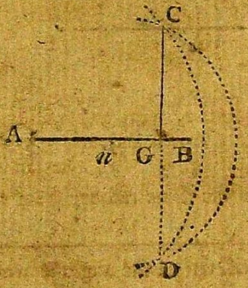
1. To any point *m*, in the line A B, draw the line C *m*.

2. Bisect the line C *m*, or divide it into two equal parts, in the point *n*.

3. From *n*, with the radius *n m*, or *n* C, describe the arc C G *m*, cutting A B in G.

4. Through the point C, draw the line C G, and it will be the perpendicular required.

Another method.



1. From

1. From A, or any other point in A B, with the radius A C, deſcribe the arc C D.

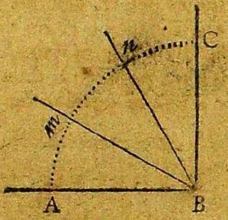
2. And from any other point *n*, in A B, with the radius *n* C, deſcribe another arc cutting the former in C, D.

3. Through the points C, D, draw the line C G D, and C G will be the perpendicular required.

N. B. Perpendiculars may be more eaſily raiſed, and let fall, in practice, by means of a ſquare, or other proper inſtrument.

PROBLEM V.

To triſect, or divide a right angle ABC into three equal parts.



1. From the point B, with any radius B A, deſcribe the arc A C, cutting the legs B A, B C, in A, C.

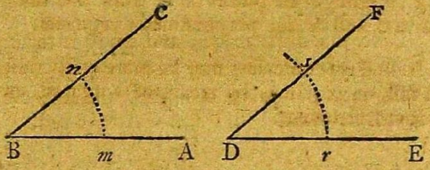
2. And from the point A, with the radius A B or B C, croſs the arc A C in *n*.

3. Alſo with the ſame radius, from the point C, croſs it in *m*.

4. Through the points *m*, *n*, draw the lines B *m*, B *n*, and they will triſect the angle as was required.

PROBLEM VI.

At a given point D *to make an angle equal to a given angle* ABC.



1. From the point B, with any radius, defcribe the arc *n m*, cutting the legs B A, B C, in the points *m*, *n*.

2. Draw the line D E, and from the point D, with the fame radius as before, defcribe the arc *r s*.

3. Take the diftance *m n*, on the former arc, and apply it to the arc *r s*, from *r* to *s*.

4. Through the points D *s*, draw the line D F, and the angle E D F will be equal to the angle A B C as was required.

PROBLEM VII.

To draw a line parallel to a given line AB.

CASE I. *When the parallel line is to pafs through a given point* C.

1. To

1. To A B, from the point C, draw any right line C *m*.

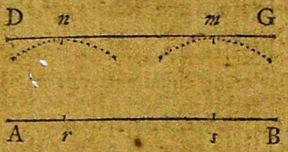
2. From the point *m*, with the radius *m* C, describe the arc C *n*, cutting A B in *n*.

3. And with the same radius, from the point C, describe the arc *m r*.

4. Take the distance C *n*, and apply it to the arc *m r*, from *m* to *r*.

5. Through the points C, *r*, draw the line G C *r* H, and it will be parallel to A B as was required.

CASE II. *When the parallel line is to be at a given distance from* A B.



1. From any two points *r*, *s*, in the line A B, with a radius equal to the given distance, describe the arcs *n*, *m*.

2. Draw the line D G, to touch those arcs without cutting them, and it will be parallel to A B as was required.

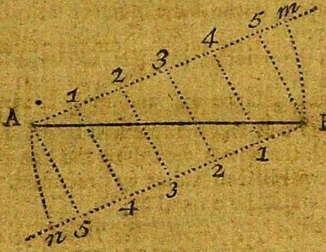
N. B. The former case of this problem, as well as several other operations in Practical Geometry, may be more easily effected by means of the parallel ruler.*

* This ruler may be had of all sizes, but is usually put into a portable case, with a drawing-pen, scale, compasses, and other useful instruments.

PRO-

PROBLEM VIII.

To divide a given line A B into any proposed number of equal parts.



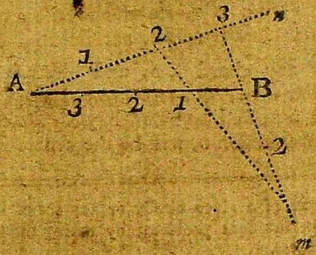
1. From one end of the line A, draw A *m*, making any angle with A B; and from B, the other end, draw B *n*, making an equal angle A B *n*.

2. In each of the lines A *m*, B *n*, beginning at A and B, set off as many equal parts, of any length, as A B is to be divided into.

3. Join the points A 5, 1 4, 2 3, &c. and A B will be divided as was required.

Note. B *n* may be drawn parallel to A *m*, by means of a parallel ruler.

Another method.



1. From

1. From the point A draw any line A n, and ſet on it as many equal parts, wanting one, as A B is to be divided into.

2. Through the points 3 B, draw the line 3 B m, and take upon it the ſame number of parts, each equal to 3 B.

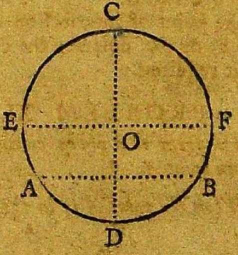
3. From the point m, in 3 B m, draw the line m 1 2, cutting A B in 1.

4. Upon A B, take the parts 1 2, 2 3, and 3 A, each equal to B 1, and the line will be divided as was required.

Note. It will be convenient in practice to draw A n ſo that B A n ſhall be a ſmall angle.

PROBLEM IX.

*To find the centre of a given circle, or one already deſcribed.**



1. Draw any chord A B, and biſect it with the perpendicular C D.

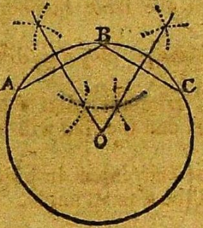
2. Biſect C D, in like manner, with the chord E F, and their interſection O, will be the centre required.

* The centre of a given circle, or any arc of it, may alſo be found as in the next problem, by taking three points in the circumference.

PRO-

PROBLEM X.

To describe the circumference of a circle through three given points, A, B, C.



1. From the middle point B, draw the lines, or chords, B A and B C.

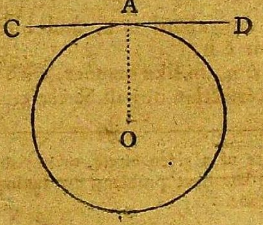
2. Bisect these chords perpendicularly, with lines meeting each other in O.

3. From the point of intersection O, with the distance O A, O B, or O C, describe the circle A B C, and it will be that required.

PROBLEM XI.

To draw a tangent to a given circle, that shall pass through a given point A.

CASE I. *When the point A is in the circumference of the circle.*

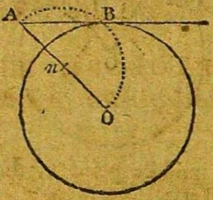


1. From

1. From the given point A, to the centre of the circle, draw the radius O A.

2. Through the point A, draw C D perpendicular to O A, and it will be the tangent required.

CASE II. *When the point A is without the circle.*



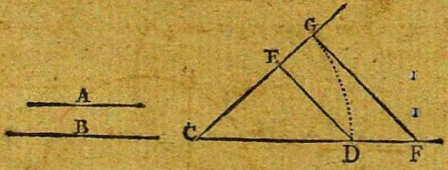
1. To the point A, from the centre O, draw the line O A, and bisect it in *n*.

2. From the point *n*, with the radius *n* A, or *n* O, describe the semi-circle A B O, cutting the given circle in B.

3. Through the points A, B, draw the line A B, and it will be the tangent required.

PROBLEM XII.

To two given right lines A, B, to find a third proportional.



1. From

1. From the point C draw two right lines, making any angle FCG.

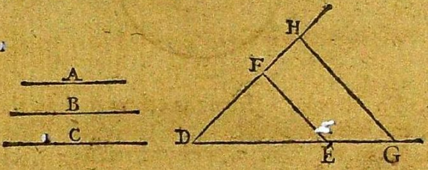
2. In these lines take CE equal to the first term A, and CG, CD, each equal to the second term B.

3. Join ED, and draw GF parallel to it; and CF will be the third proportional required.

That is CE (A) : CG (B) :: CD (B) : CF.

PROBLEM XIII.

To three given right lines A, B, C, to find a fourth proportional.



1. From the point D draw two right lines, making any angle GDH.

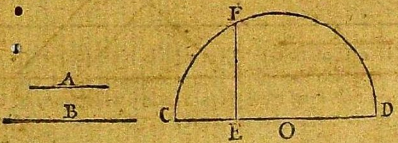
2. In these lines take DF equal to the first term A, DE equal to the second B, and DH equal to the third C.

3. Join FE, and draw HG parallel to it, and DG will be the fourth proportional required.

That is DF (A) : DE (B) :: DH (C) : DG.

PROBLEM XIV.

Between two given right lines A, B, to find a mean proportional.



1. Draw

1. Draw any right line, in which take C E equal to A, and E D equal to B.

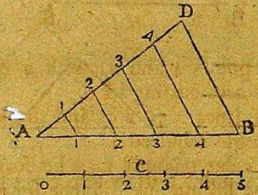
2. Biſect C D, in O, and with O D or O C, as radius, deſcribe the ſemi-circle C F D.

3. From the point E draw E F perpendicular to C D, and it will be the mean proportional required.

That is C E (A) : E F :: E F : E D (B).

PROBLEM XV.

*To divide a given line A B in the ſame proportion that another given line C is divided.**



1. From the point A draw A D equal to the given line C, and making any angle with A B.

2. To A D apply the ſeveral diviſions of C, and join D B.

3. Draw the lines 44, 33, &c. each parallel to D B, and the line A B will be divided as was required.

That is the parts A 1, 1 2, 2 3, 3 4, 4 B, on the line A B, will be proportional to the parts O 1, 1 2, 2 3, 3 4, 4 5, on the line C.

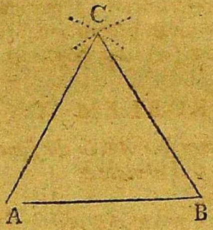
---

* The caſe of this problem which moſt frequently occurs, is that in which the given line is required to be divided into two parts that ſhall have given a ratio; which may be done in nearly the ſame manner as above.

PRO-

PROBLEM XVI.

Upon a given right line AB, to make an equilateral triangle.



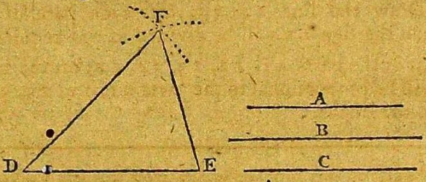
1. From the points A and B, with a radius equal to A B, describe arcs cutting in C.

2. Draw the lines A C, B C, and the figure A C B will be the triangle required.

Note. An isosceles triangle may be formed in the same manner, by taking any distance for radius.

PROBLEM XVII.

*To make a triangle whose three sides shall be respectively equal to three given lines A, B, C.**



* The three given lines must be each of such a length that any two of them taken together shall be greater than the third.

1. Draw

1. Draw a line D E equal to one of the given lines C.

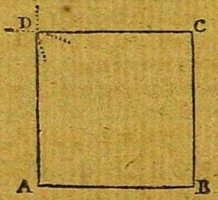
2. On the point D, with a radius equal to B, describe an arc.

3. And on the point E, with a radius equal to A, describe another arc, cutting the former in F.

4. Draw the lines D F, E F, and D F E will be the triangle required.

PROBLEM XVIII.

Upon a given line A B to describe a square.



1. From the point B, draw B C perpendicular, and equal to A B.

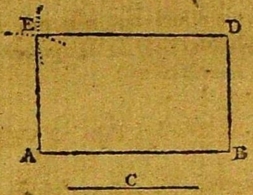
2. On A and C, with the radius A B, describe two arcs cutting each other in D.

3. Draw the lines A D, C D, and the figure A B C D will be the square required.

Note. A rhombus may be made on the given line A B in exactly the same manner, if B C be drawn with the proper obliquity, instead of perpendicularly.

PROBLEM XIX.

To describe a rectangle, whose length and breadth shall be equal to two given lines AB and C.



1. At the point B, in the given line A B, erect the perpendicular B D, and make it equal to C.

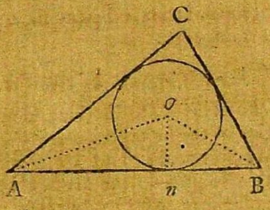
2. From the points D, A, with the radii A B and C, describe two arcs cutting each other in E.

3. Join E A and E D, and A B D E will be the rectangle required.

Note. A parallelogram may be described in nearly the same manner.

PROBLEM XX.

In a given triangle ABC to inscribe a circle.

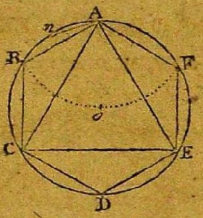


1. Biſect the angles A and B with the lines A O and B O.

2. From the point of interſection O let fall the perpendicular O n, and it will be the radius of the circle required.

PROBLEM XXI.

In a given circle to inſcribe an equilateral triangle, an hexagon, or a dodecagon.



For the hexagon.

1. From any point A as a centre, with a diſtance equal to the radius A o, deſcribe the arc B o F.

2. Join the points A B, or A F, and either of theſe lines being carried ſix times round the circle will form the hexagon required.

That is, the radius of the circle is equal to the ſide of the hexagon; and the ſides of the hexagon divide the circumference of the circle into ſix equal parts, each containing 60 degrees.

For the equilateral triangle.

1. From the point A, to the ſecond and fourth diviſions, or angles of the hexagon, draw the lines A C, A E.

2. Join the points C E, and A C E will be the equilateral triangle required; the arc A C being one third of the circumference, or 120 degrees.

 For

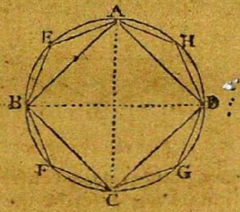
For the dodecagon.

Bisect the arc A B of the hexagon in the point *n*, and the line A *n* being carried twelve times round the circumference, will form the dodecagon required, the arc A *n* being 30 degrees.

If A *n* be again bisected, a polygon may be formed of 24 sides; and by another bisection a polygon of 48 sides; and so on.

PROBLEM XXII.

To describe a square, or an octagon, in a given circle.



For the square.

1. Draw the diameters B D and A C, intersecting each other at right angles.
2. Join the points A B, B C, C D, and D A, and A B C D will be the square required.

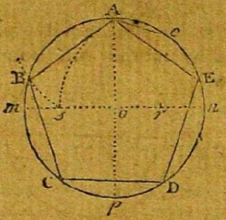
For the octagon.

Bisect the arc A B of the square in the point E, and the line A E being carried eight times round the circumference, will form the octagon.

If the arc A E be again bisected, a polygon may be formed of 16 sides; and by another bisection a polygon of 32 sides; and so on.

PROBLEM XXIII.*

To inscribe a pentagon, or decagon, in a given circle.



For the pentagon.

1. Draw the diameters A *p*, *n m* at right angles to each other, and bisect the radius o *n* in *r*.

2. From the point *r*, with the distance *r* A, describe the arc A *s*, and from the point A, with the distance A *s*, describe the arc *s* B.

3. Join the points A, B, and the line A B being carried five times round the circle, will form the pentagon required.

For the decagon.

Bisect the arc A E of the pentagon in *c*, and the line A *c* being carried ten times round the circumference will form the decagon required.

If the arc A *c* be again bisected, a polygon of 20 sides may be formed; and by another bisection, a polygon of 40 sides; and so on.

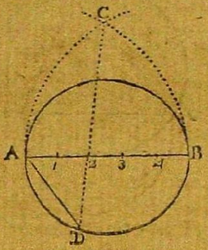
---

* Besides the figures here constructed, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, from any known method *purely geometrical.*

PRO-

PROBLEM XXIV.*

In a given circle to inscribe any regular polygon.



1. Draw the diameter A B, which divide into as many equal parts as the figure has sides.

2. From the points A B, as centres, with the radius A B, describe arcs crossing each other in C.

3. From the point C, through the second division of the diameter, draw the line C D.

4. Join the points A, D, and the line A D will be the side of the polygon required.

Note. In this construction A D is the side of a pentagon.

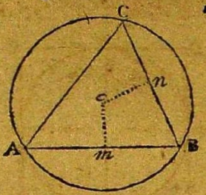
Another method, something more accurate, is by erecting a perpendicular from the centre, of such a length that the part without the circle shall be equal to $\frac{3}{4}$ of that within, and drawing a line from its extremity through the second division as before.

---

* This construction is the invention of *Renaldinus,* and was first given in his 2d Book *De Resol. &c. Comp. Mathem.* page 367. The rule for polygons in general is only an approximation, but holds true in the equilateral triangle and hexagon.

PROBLEM XXV.

About a given triangle ABC to circumſcribe a circle.



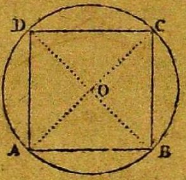
1. Biſect the two ſides A B, B C with the perpendiculars *m o* and *n o*.

2. From the point of interſection *o*, with the diſtance O A, O B or O C, deſcribe the circle A C B, and it will be that required.

If any two of the angles be biſected, inſtead of the ſides, the interſection of the lines will alſo give the centre of the circle.

PROBLEM XXVI.

About a given ſquare ABCD to circumſcribe a circle.



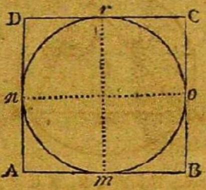
1. Draw the two diagonals A C and B D, interſecting each other in O.

2. From the point O, with the diſtance O A, O B, O C, or O D, deſcribe the circle A B C D, and it will be that required.

PRO-

PROBLEM XXVII.

To circumscribe a square about a given circle.

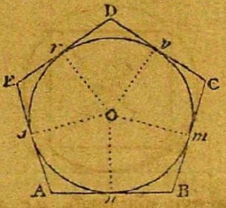


1. Draw any two diameters *n o* and *r m* at right angles to each other.

2. Through the points *m o r n*, draw the lines A B, B C, C D, and D A, perpendicular to *r m* and *n o*, and A B C D will be the square required.*

PROBLEM XXVIII.

About a given circle to circumscribe a pentagon.



* If each of the quadrants *rn*, *nm*, *mo* and *or* be bisected, and tangents be drawn to those points, the circumscribing figure will be an octagon.

 1. Inscribe

1. Inſcribe a pentagon in the circle; or, which is the ſame thing, find the points *n*, *m*, *v*, *r*, *s*, as in Problem XXIII.

2. From the centre *o*, to each of theſe points, draw the radii *on*, *om*, *ov*, *or* and *os*.

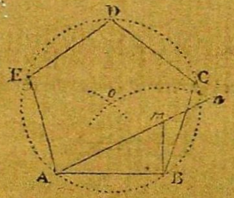
3. Through the points *n*, *m*, draw the lines A B, B C perpendicular to *on*, *om*; producing them till they meet each other at B.

4. In the ſame manner, draw the lines C D, D E, E A, and A B C D E will be the pentagon required.

Note. Any other polygon may be made to circumſcribe a circle, by firſt inſcribing a ſimilar one, and then drawing tangents to the circle at the angular points.

PROBLEM XXIX.*

On a given line A B *to make a regular pentagon.*



1. Make B *m* perpendicular to A B, and equal to one half of it.

---

* In the former edition of this work, another method of deſcribing a pentagon was given, as firſt propoſed by *Albertus Durer*, in his Geometry, p. 55, printed 1532; but as that is only an approximation, and is not more eaſy in practice than the preſent one, which is perfectly accurate, it is here omitted.

2. Draw

2. Draw A *m*, and produce it till the part *m n* is equal to B *m*.

3. From A and B as centres, with the radius B *n*, describe arcs cutting each other in *o*.

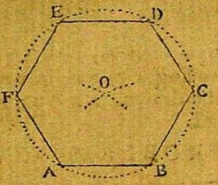
4. And from the point *o*, with the same radius, or with *o* A, or *o* B, describe the circle A B C D E.

5. Apply the line A E five times round the circumference of this circle, and it will form the pentagon required.

Note. If tangents be drawn through the angular points A, B, C, D, E a pentagon circumscribing the circle will be formed; and if the arcs be bisected, a circumscribing decagon may be formed.

PROBLEM XXX.

On a given line A B *to make a regular hexagon.*



1. From the points A, B as centres, with the radius A B, describe arcs cutting each other in *o*.

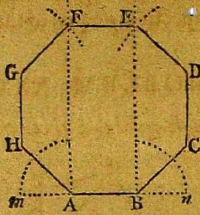
2. And from the point O, with the distance O A or O B, describe the circle A B C D E F.

3. Apply the line A B six times round the circumference, and it will form the hexagon required.*

* This construction is founded on the principle, that the radius of every circle is equal to the side of its inscribed hexagon, or the chord of 60°.

PROBLEM XXXI.

On a given line A B *to form a regular octagon.*



1. On the extremes of the given line A B erect the indefinite perpendiculars A F and B E.

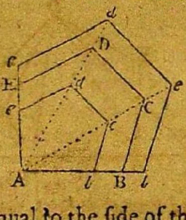
2. Produce A B both ways to *m* and *n*, and bisect the angles, *m* A F and *n* B E with the lines A H and B C.

3. Make A H and B C each equal to A B, and draw H G, C D parallel to A F or B E, and also each equal to A B.

4. From G, D, as centres, with a radius equal to A B, describe arcs crossing A F, B E, in F and E; and if G F, F E, and E D, be drawn, A B C D E F G H will be the octagon required.

PROBLEM XXXII.

To make a figure similar to a given figure A B C D E.



1. Take A *b* equal to the side of the figure required, and from the angle A draw the diagonals, A C, A D.

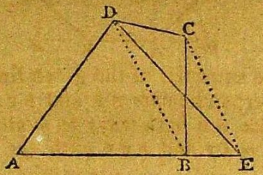
2. From

2. From the points *b*, *c*, *d* draw *bc*, *cd*, *de* parallel to B C, C D, D E, and *Abcde* will be similar to A B C D E.

The same thing may also be done by making the angles *b*,*c*,*d*,*e* respectively equal to the angles B, C, D, E.

PROBLEM XXXIII.

To make a triangle equal to a given trapezium A B C D.

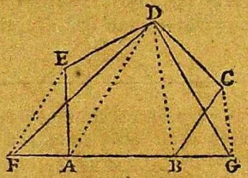


1. Draw the diagonal D B, and make C E parallel to it, meeting the side A B produced in E.

2. Join the points D, E, and A D E will be the triangle required.

PROBLEM XXXIV.

To make a triangle equal to any right lined figure A B C D E A.



1. Produce the side A B both ways at pleasure.

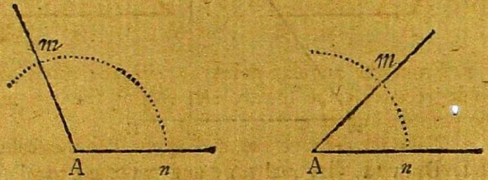
2. Draw the diagonals D A, D B, and parallel to them the lines E F and C G.

3. Join the points D F, D G, and D F G will be the triangle required.

And in nearly the same manner may any right lined figure whatever be reduced to a triangle.

PROBLEM XXXV.*

To make an angle of any proposed number of degrees.



1. Take the first 60 degrees from the scale of chords, and from the point A, with this radius, describe the arc *n m*.

2. Take the chord of the proposed number of degrees from the same scale, and apply it from *n* to *m*.

3. From the point A draw the lines A *n* and A *m*, and they will form the angle required.

Note. Angles greater than 90° are usually taken at twice.

---

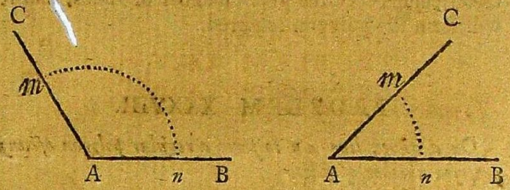
* The line of chords made use of in the following problems, is commonly put upon the plain scale, and is adapted to 90 degrees, or the fourth part of a circle.

For a description of this, and other instruments made use of in Practical Geometry, see *Mr. Robertson's Treatise on such mathematical instruments as are usually put into a portable case.*

PRO-

PROBLEM XXXVI.

Any angle B A C *being given, to find the number of degrees it contains.**



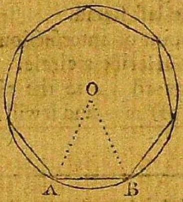
1. From the angular point A, with the chord of 60 degrees, describe the arc *nm*, cutting the legs in the points *n* and *m*.

2. Take the distance *n m*, and apply it to the scale of chords, and it will shew the degrees required.

Note. When the distance *nm* is greater than 90°, it must be taken at twice, as before.

PROBLEM XXXVII.

In a given circle to inscribe a polygon of any proposed number of sides.



* Both this and the last problem may be performed by means of a *protractor*, which is a graduated arc designed for that purpose.

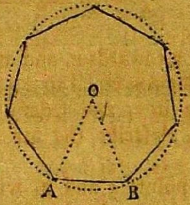
1. Divide

1. Divide 360° by the number of ſides, and make an angle A O B, at the centre, whoſe meaſure ſhall be equal to the degrees in the quotient.

2. Join the points A B, and apply the chord A B to the circumference the given number of times, and it will form the polygon required.

PROBLEM XXXVIII.

On a given line A B to form a regular polygon of any propoſed number of ſides.



1. Divide 360° by the number of ſides, and ſubtract the quotient from 180 degrees.

2. Make the angles A B O and B A O each equal to half the difference laſt found.

3. From the point of interſection O, with the diſtance O A or O B, deſcribe a circle.

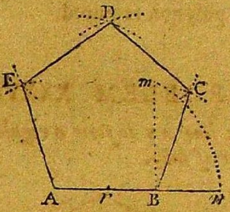
4. Apply the chord A B to the circumference the propoſed number of times, and it will form the polygon required.*

* By this method the circumference of a circle may alſo be divided into any number of equal parts; for if 360° be divided by the number of parts, and the angle A O B be made equal to the degrees in the quotient, the arc A B will be one of the equal parts required.

PRO-

PROBLEM XXXIX.*

Upon a given right line A B *to defcribe a regular pentagon.*



1. Produce A B towards *n*, and at the point B make the perpendicular B *m* equal to A B.

2. Bifect A B in *r*, and from *r* as a centre, with the radius *r m*, defcribe the arc *m n*; cutting A B in *n*.

3. From the points A and B, with the radius A *n*, defcribe arcs cutting each other in D.

4. And from the points A, D and B, D, with the radius A B, defcribe arcs cutting each other in C and E.

5. Join B C, D C, D E and E A, and A B C D E will be the pentagon required.

This method differs but little from that of Problem XXIX, and is equally eafy and convenient in practice.

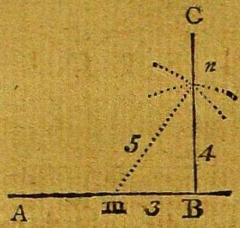
---

* This and the following problem were not given in the firft Edition of this work, but are now added on account of their elegance and utility. The fecond is derived from the 47th Prop. B. I. Euclid's Elements, and the firft is propofed for a demonftration in the Ladies Diary for the year 1786.

PRO-

PROBLEM XL.

To raiſe a perpendicular from any point B *in a given line* AB.



1. From any ſcale of equal parts take a diſtance equal to 3 diviſions, and ſet it from B to *m*.

2. And from the points B and *m*, with the diſtances 4 and 5, taken from the ſame ſcale, deſcribe arcs cutting each other in *n*.

3. Through the points *n*, B, draw the line B C, and it will be the perpendicular required.

Explanation of the characters made uſe of in the following part of the Work.

$+$ Is the ſign of addition.
$-$ ——— of ſubtraction.
$\times$ ——— of multiplication.
$\div$ ——— of diviſion.
$\surd$ ——— of the ſquare root.
$\sqrt[3]{}$ ——— of the cube root.
$=$ ——— of equality.
$: \,::\, :$ ——— of proportion.

OF THE

MENSURATION

OF

SUPERFICIES.

THE *area* of any figure is the meaſure of its ſurface, or the ſpace contained within the bounds of that ſurface, without any regard to thickneſs.

A ſquare whoſe ſide is one inch, one foot, or one yard, &c. is called the *meaſuring unit*, and the area or content of any figure is computed by the number of thoſe ſquares contained in that figure.

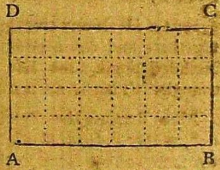
PROBLEM I.

To find the area of a parallelogram; whether it be a ſquare, a rectangle, a rhombus, or a rhomboides.

RULE.*

Multiply the length by the perpendicular height, and the product will be the area.

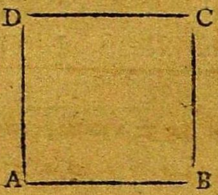
EXAM-



* Take any rectangle ABCD, and divide each of its ſides, reſpectively, into as many equal parts as is expreſſed by the number of times they contain the linear meaſuring unit, and let all the oppoſite points of diviſion be connected by right lines. Then, it is evident, that theſe lines divide the rectangle

EXAMPLES.

1. Required the area of the ſquare ABCD whoſe ſide is 5 feet 9 inches.



Here 5 *fe.* 9 *in.* = 5·75; *and* $\overline{5\cdot75}|^2 = 5\cdot75 \times 5\cdot75 =$ 33·0625 *feet* = 33 *fe.* 0 *in.* 9 *pa.* = *area required.*

---

rectangle into a number of ſquares each equal to the ſuperficial meaſuring unit, and that the number of theſe ſquares, or the area of the figure, is equal to the number of linear meaſuring units in the length, as often repeated as there are linear meaſuring units in the breadth or height, that is equal to the length multiplied by the height, *which is the rule.*

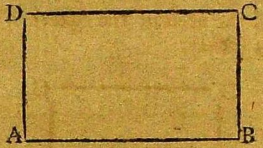
And ſince a rectangle is equal to an oblique parallelogram ſtanding upon the ſame baſe, and between the ſame parallels, (Euc. I. 35,) the rule is true for any parallelogram in general. Q. E. D.

RULE II. If any two ſides of a parallelogram be multiplied together, and the product again by the natural ſine of their included angle, the laſt product will give the area of the triangle. That is AB × BC × nat. ſine of the angle B = area.

Note. Becauſe the angles of a ſquare and rectangle are each 90°, whoſe natural ſine is unity, or 1, the rule in this caſe is the ſame as that given in the text.

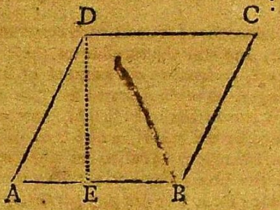
2. Re-

2. Required the area of the rectangle A B C D, whose length A B is 13.75 chains, and breadth B C 9.5 chains.



Here $13.75 \times 9.5 = 130.625$; *and* $\frac{130.625}{10} =$ 13.0625 *acres* = 13 *ac.* 0 *ro.* 10 *po.* = *area required.*

3. Required the area of the rhombus A B C D, whose length A B is 12 *fe.* 6 *in.* and its height D E 9 *fe.* 3 *in.*

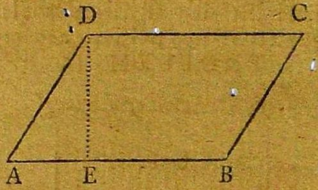


Here 12 *fe.* 6 *in.* = 12.5, *and* 9 *fe.* 3 *in.* = 9.25;

Whence $12.5 \times 9.25 = 115.625$ *feet.* = 115 *fe.* 7 *in.* 6 *pa.* = *area required.*

4. What is the area of the rhomboides A B C D, whose length A B is 10.52 chains, and height D E 7.63 chains?

Here



Here $10.52 \times 7.63 = 80.2676$; *and* $\frac{80.2676}{10} = 80.2676$ *acres* = 8 *ac.* 0 *ro.* 4 *po.* = *area required.*

5. What is the area of a ſquare whoſe ſide is 35.25 chains?

ac. ro. po.
Anſ. 124 1 1

6. What is the area of a ſquare whoſe ſide is 8 feet 4 inches?

fe. in. pa.
Anſ. 69 5 4

7. What is the area of a rectangle whoſe length is 14 feet 6 inches, and breadth 4 feet 9 inches?

fe. in. pa.
Anſ. 68 10 6

8. Required the area of a rhombus, the length of whoſe ſide is 12.24 feet, and height 9.16 feet.

fe. in. pa.
Anſ. 112 1 5

9. Required the area of a rhomboides whoſe length is 10.51 chains, and breadth 4.28 chains.

ac. ro. p.
Anſ. 4 1 39

10. What is the area of a rhomboides whoſe length is 7 feet 9 inches, and height 3 feet 6 inches?

fe. in. pa.
Anſ. 17 1 6

11. To find the area of a rectangular board, whoſe length is $12\frac{1}{2}$ feet, and breadth 9 inches.

Anſ. $9\frac{3}{8}$ *feet.*

PRO-

PROBLEM II.

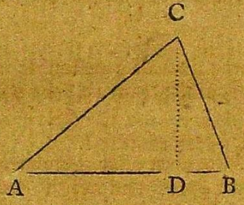
To find the area of a triangle.

RULE.*

Multiply the base by the perpendicular height, and half the product will be the area.

EXAMPLES.

1. Required the area of the triangle ABC, whose base AB is 10 feet 9 inches, and height DC 7 feet 3 inches.



Here 10 *fe.* 9 *in.* = 10.75, *and* 7 *fe.* 3 *in.* = 7.25;

Whence $10.75 \times 7.25 = 77.9375$, *and* $\frac{77.9375}{2} =$

38.96875 *feet* = 38 *fe.* 11 *in.* $7\frac{1}{2}$ *pa.* = *area required.*

2. What is the area of a triangle whose base is 18 feet 4 inches, and height 11 feet 10 inches?

fe. in. pa.
Ans. 108 5 8

---

* A triangle is half a parallelogram of the same base and altitude, (Euc. I. 41.) and therefore the truth of the rule is evident.

3. What

3. What is the area of a triangle whose base is 16.75 feet, and height 6.24 feet? *fe. in. pa.* *Ans.* 52 3 1

4. Required the area of a triangle whose base is 12.25 chains, and height 8.5 chains. *ac. ro. po.* *Ans.* 5 0 33

5. What is the area of a triangle whose base is 20 feet, and height 10.25? *Ans.* 102.5 *fe.*

PROBLEM III.

To find the area of a triangle whose three sides only are given *.

RULE.

1. From half the sum of the three sides subtract each side severally.

2. Multiply the half sum and the three remainders continually together, and the square root of the product will be the area required.

EXAM-

---

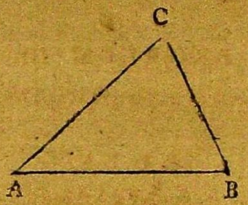
* *Demon.* Let AC $= a$, AB $= b$, BC $= c$, and AD $= x$: (See preceding fig.) Then, since BD $= b - x$, we shall have $c^2 - (b-x)^2 = \text{CD}^2 = a^2 - x^2$, or $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$ from which x is found $= \frac{a^2+b^2-c^2}{2b}$ by transf. and reduction.

But $\text{CD}^2 = \text{AC}^2 - \text{AD}^2 = \overline{\text{AC}+\text{AD}} \times \overline{\text{AC}-\text{AD}} = (a + \frac{a^2+b^2-c^2}{2b}) \times (a - \frac{a^2+b^2-c^2}{2b}) = \frac{2ab+a^2+b^2-c^2}{2b} \times \frac{2ab-a^2-b^2+c^2}{2b} = \frac{(a+b)^2-c^2}{2b} \times \frac{c^2-(a-b)^2}{2b}$;

Whence CD $= \frac{1}{2b}\sqrt{(\overline{a+b}^2-c^2) \times (c^2-\overline{a-b}^2,)}$ and the

EXAMPLES.

1. Required the area of the triangle A B C, whose three sides B C, C A and A B, are 24, 36, and 48 chains respectively.



Here $\frac{24+36+48}{2} = \frac{108}{2} = 54 = \frac{1}{2}$ *sum of the sides*;

Also $54-24=30$ *first diff.* $54-36=18$ *second diff.* *and* $54-48=6$ *third diff.*

Whence $\sqrt{54 \times 30 \times 18 \times 6} = \sqrt{174960} = 418.043 =$ *area required.*

2. Re-

---

the area $\frac{1}{2}$ AB $\times$ CD $= \frac{1}{4}\sqrt{:(\overline{a+b}^2 - c^2) \times (c^2 - \overline{a-b}^2)}$ $= \frac{1}{4}\sqrt{(\overline{a+b+c} \times \overline{a+b-c} \times \overline{c+a-b} \times \overline{c-a+b})}$ $= \sqrt{(\frac{a+b+c}{2} \times \frac{a+b-c}{2} \times \frac{c+a-b}{2} \times \frac{c-a+b}{2},)}$ which, by making $s = \frac{1}{2} \times (a+b+c)$ becomes $= \sqrt{(s \times \overline{s-c} \times \overline{s-b} \times \overline{s-a})} =$ algebraical expression for the rule, as was to be demonstrated.

Cor. 1. If s be put equal to $a+b$ and $d = b \backsim c$, the rule is $\sqrt{(s^2-a^2) \times (a^2-d^2)}$.

Cor.

2. Required the area of a triangle whose three sides are 13, 14 and 15 feet. *Ans.* 84 *square feet.*

3. How many acres are there in a triangle whose three sides are 49.00, 50.25 and 25.69 chains?

Ans. 61.498 *ac.*

4. Required the area of a right angled triangle, whose hypothenuse is 50, and the other two sides 30 and 40. *Ans.* 600.

5. Required the area of an equilateral triangle whose side is 25. *Ans.* 270.625.

6. Required the area of an Isosceles triangle whose base is 20, and each of its equal sides 15.

Ans. 111.803.

7. Required the area of a triangle whose three sides are 20, 30 and 40 chains.

Ans. 29 *ac.* 7 *po.*

PROBLEM IV.

Any two sides of a right angled triangle being given to find the third side.

Cor. 2. If all the sides be equal, the rule will become $\frac{1}{4}a^2\sqrt{3}$, or $\frac{1}{4}a^2 \times 1.732$ for the equilateral triangle whose side is a.

Cor. 3. If the triangle be right angled, a being the hypothenuse, the rule will be $\frac{a+b+c}{2} \times \frac{-a+b+c}{2}$, or $\frac{1}{2}p \times \frac{1}{2}p - a$, putting p for the perimeter.

Rule II. Any two sides of a triangle being multiplied together, and the product again by half the natural sine of their included angle, will give the area of the triangle.

That is, AC × CB × nat. sine of the angle C = twice area.

RULE.

RULE.*

1. *When the two legs are given, to find the hypothenuſe.*

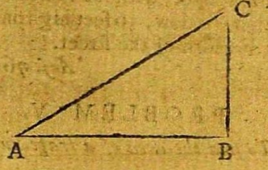
Add the ſquare of one of the legs to the ſquare of the other, and the ſquare root of the ſum will be equal to the hypothenuſe.

2. *When the hypothenuſe and one of the legs are given, to find the other leg.*

From the ſquare of the hypothenuſe take the ſquare of the given leg, and the ſquare root of the remainder will be equal to the other leg.

EXAMPLES.

1. In the right angled triangle ABC, the baſe AB is 56, and the perpendicular BC 33: what is the hypothenuſe?



Here $56^2+33^2=3136+1089=4225$; *and* $\sqrt{4225}=65=$ *hypothenuſe* AC.

2. If the hypothenuſe AC be 53, and the baſe AB 45: what is the perpendicular BC?

* By Euc. 47. I. $\overline{AB^2+BC^2}=AC^2$, or $\overline{AC^2-AB^2}=BC$; and therefore $\sqrt{AB^2+BC^2}=AC$; or $\sqrt{AC^2-AB^2}=BC$, or $\sqrt{AC^2-BC^2}=AB$ which *is the ſame as the rule.*

Here

Here $53^2 - 45^2 = 2809 - 2025 = 784$; *and* $\sqrt{784} = 28 =$ *perpendicular* BC.

3. The base of a right angled triangle is 77, and the perpendicular 36: what is the hypothenuse?

Ans. 85.

4. The hypothenuse of a right angled triangle is 109, and the perpendicular 60: what is the base?

Ans. 91.

5. It is required to find the length of a shoar, which strutting 12 feet from the upright of a building, will support a jaumb 20 feet from the ground.

Ans. 23.32380 *feet.*

6. The height of a precipice, standing close by the side of a river, is 103 feet, and a line of 320 feet will reach from the top of it to the opposite bank; required the breadth of the river. *Ans.* 302.9703 *feet.*

7. A ladder 50 feet long, being placed in a street, reached a window 28 feet from the ground, on one side; and by turning it over, without removing the foot, it reached another window, 36 feet high on the other side: required the breadth of the street.

Ans. 76.1233335 *feet.*

PROBLEM V.

To find the area of a trapezium.

RULE.*

Multiply the diagonal by the sum of the two perpendiculars falling upon it from the opposite angles, and half the product will be the area.

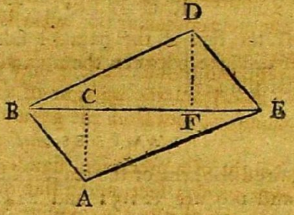
EXAM-

---

* *Demon.* The area of the triangle BDE is $= \frac{BE \times DF}{2}$; and the area of the triangle BAE is $= \frac{BE \times AC}{2}$; and there-

EXAMPLES.

1. Required the area of the trapezium BAED, whose diagonal BE is 84, the perpendicular AC 21, and DF 28.



Here $\overline{28+21}\times 84 = 49\times 84 = 4116$; *and* $\frac{4116}{2} =$ 2058 *the area required.*

2. Required the area of a trapezium whose diagonal is 80.5, and the two perpendiculars 24.5 and 30.1.

Ans. 2197.65.

3. What is the area of a trapezium whose diagonal is 108 feet 6 inches, and the perpendiculars 56 feet 3 inches, and 60 feet 9 inches? *Ans.* 6347 *fe.* 3 *in.*

---

therefore the sum of these areas, or the area of the whole trapezium, is $= \frac{BE\times DF}{2} + \frac{BE\times AC}{2} = \frac{DF+AC}{2}\times BE$. Q.E.D.

If the trapezium can be inscribed in a circle; that is, if the sum of two of its opposite angles is equal to two right angles, or 180°, the area may be found thus:

Rule. From half the sum of the four sides subtract each side severally; then multiply the four remainders continually together, and the square root of the product will be the area.

PROBLEM VI.

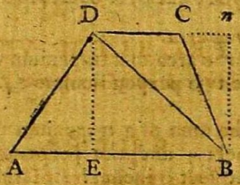
To find the area of a trapezoid, or a quadrangle, two of whose opposite sides are parallel.

RULE.*

Multiply the sum of the parallel sides by the perpendicular distance between them, and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezoid ABCD, whose sides AB and DC are 321.51 and 214.24, and perpendicular DE 171.16.



Here 321.51 + 214.24 = 535.75 = *sum of the parallel sides* AB, DC.

---

* *Demon.* The △ ABD is $= \frac{AB \times DE}{2}$, and the △ BCD $= \frac{DC \times Bn}{2}$, or (because $Bn = DE$) $= \frac{DC \times DE}{2}$. Therefore △ ABD + △ BCD, or the whole trapezoid ABCD, is $= \frac{AB \times DE}{2} + \frac{DC \times DE}{2} = \frac{AB + DC}{2} \times DE$. Q.E.D.

Whence

Whence 535.75 × 171.16 (*the perp.* DE) = 91698.9700;

And $\frac{91698.9700}{2}$ = 45849.485 *the area required.*

2. The parallel ſides of a trapezoid are 12.41 and 8.22 chains, and the perpendicular diſtance 5.15 chains; required the area. *ac. ro. po.*

Anſ. 5 1 9

3. Required the area of a trapezoid whoſe parallel ſides are 25 feet 6 inches and 18 feet 9 inches, and the perpendicular diſtance 10 feet 5 inches? *fe. in. pa.*

Anſ. 230 5 7

4. Required the area of a trapezoid whoſe parallel ſides are 20.5 and 12.25, and perpendicular diſtance 10.75. *Anſ.* 176.03125.

PROBLEM VII.

To find the area of a regular polygon.

RULE*.

Multiply half the perimeter of the figure by the perpendicular falling from its centre upon one of the ſides, and the product will be the area.

Note. The perimeter of any figure is the ſum of all its ſides.

EXAM-

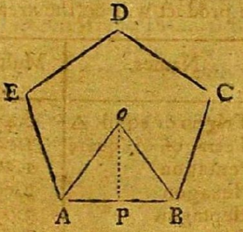
---

* *Demon.* Every regular polygon is compoſed of as many equal triangles as it has ſides, conſequently the area of one of thoſe triangles being multiplied by the number of ſides muſt give the area of the whole figure; but the area of either of the triangles is equal to the rectangle of the perpendicular and half the baſe, and therefore the rectangle of the perpendicular and

D 3

EXAMPLES.

1. Required the area of the regular pentagon ABCDE whose side AB, or BC, &c. is 25 feet, and perpendicular OP 17.2 feet?



Here $\frac{25 \times 5}{2} = 62.5 =$ *half perimeter; and* $62.5 \times 17.2 = 1075$ *square feet = area required.*

2. Required the area of a hexagon whose side is 14.6 feet, and perpendicular 12.64 feet?

Ans. 553.632 *square feet.*

3. Required the area of a heptagon whose side is 19.38, and perpendicular from the centre 20?

Ans. 1356.6.

4. Required the area of an octagon whose side is 9.941, and perpendicular 12? *Ans.* 477.168.

---

and half the sum of the sides is equal to the area of the whole polygon: thus, $OP \times \frac{AB}{2}$ is = area of the △ AOB, and $OP \times \frac{5AB}{2}$ = area of the polygon ABCDE. Q.E.D.

PRO-

PROBLEM VIII.

To find the area of a regular polygon, when the side only is given.

RULE.*

Multiply the square of the side of the polygon, by the number standing opposite to its name in the following table, and the product will be the area.

N° of sides.	Names.	Multipliers.
3	Trigon or equil. △	0.433013 −
4	Tetragon or square	1.000000 +
5	Pentagon	1.720477 +
6	Hexagon	2.598076 +
7	Heptagon	3.633912 +
8	Octagon	4.828427 +
9	Nonagon	6.181824 +
10	Decagon	7.694209 −
11	Undecagon	9.365640 −
12	Duodecagon	11.196152 +

EXAM-

* *Demon.* The multipliers in the table are the areas of the polygons to which they belong when the side is unity or 1.

Whence as all regular polygons, of the same number of sides, are similar to each other, and as similar figures are as the squares of their like sides, (*Euc.* VI. 20.) 1^2 : multiplier in the table :: square of the side of any polygon : area of that polygon; or, which is the same thing, the square of the side of any polygon × by its tabular number is = area of the polygon. Q. E. D.

The table is formed by trigonometry thus: As radius = 1 : tang. ∠ OBP :: BP ($\frac{1}{2}$) : PO = $\frac{\text{BP} \times \text{tang.} \angle \text{OBP}}{\text{radius.}}$ = $\frac{1}{2}$ tang. ∠ OBP; whence OP × BP = $\frac{1}{4}$ tang. ∠ OBP = area of

EXAMPLES.

1. Required the area of a pentagon whose side is 15.

$$\begin{array}{r} 15 \\ 15 \\ \hline 75 \\ 15 \\ \hline 225 \end{array} = \textit{square of the side.}$$

$$\begin{array}{r} 1.720477 \\ 225 \\ \hline 8602385 \\ 3440954 \\ 3440954 \\ \hline 387.107325 \end{array}$$

1.720477 = *area, when the side is* 1.

387.107325 = *area required.*

---

of the △ AOB; and ½ tang. ∠ OBP × number of sides = tabular number, or the area of the polygon.

The angle OBP, together with its tangent, for any polygon of not more than 12 sides, is shewn in the following table.

No of sides	Names.	Angle OBP	Tangents.
3	Trigon	30°	$\cdot57735+ = \frac{1}{3}\sqrt{3}$
4	Tetragon	45°	$1.00000+ = 1\times1$
5	Pentagon	54°	$1\cdot37638+ = \sqrt{1+\frac{2}{5}\sqrt{5}}$
6	Hexagon	60°	$1.73205+ = \sqrt{3}$
7	Heptagon	$64^\circ\frac{2}{7}$	$2.07652+$
8	Octagon	$67^\circ\frac{1}{2}$	$2.41421+ = 1+\sqrt{2}$
9	Nonagon	70°	$2.74747+$
10	Decagon	72°	$3.07768+ = \sqrt{5+2\sqrt{5}}$
11	Undecagon	$73^\circ\frac{7}{11}$	$3\cdot40568+$
12	Duodecagon	75°	$3\cdot73205+ = 2+\sqrt{3}$

2. The

2. The ſide of a hexagon is 5 feet 4 inches; what is the area? *Anſ.* 73.9.

3. Required the area of an octagon whoſe ſide is 16? *Anſ.* 1236.0773.

4. Required the area of a decagon whoſe ſide is 20.5? *Anſ.* 3233.4913.

PROBLEM IX.

The diameter of a circle being given to find the circumference; or, the circumference being given to find the diameter.

RULE I.*

As 7 is to 22, ſo is the diameter to the circumference. Or, as 22 is to 7, ſo is the circumference to the diameter.

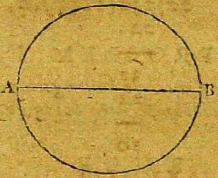
EXAM-

* The proportion of the diameter of a circle to its circumference has never yet been exactly attained. Nor can a ſquare, or any other right lined figure, be found, that ſhall be equal to a given circle. This is the celebrated problem called the *ſquaring of the circle*, which has exerciſed the abilities of the greateſt mathematicians for ages, and been the occaſion of ſo many endleſs diſputes. Several perſons of conſiderable eminence have, at different times, pretended that they had diſcovered the exact quadrature; but their errors have ſoon been detected, and it is now generally looked upon as a thing impoſſible to be done.

But though the relation between the diameter and circumference cannot be accurately expreſſed in known numbers, it may yet be approximated to any aſſigned degree of exactneſs. And in this manner was the problem ſolved by the great *Archimedes*, about two thouſand years ago, who diſcovered the proportion to be nearly as 7 is to 22; *which is the ſame as our firſt rule.* This he effected by ſhewing that the perimeter of a circumſcribed regular polygon of 192 ſides is to the diameter in a leſs ratio

EXAMPLES.

1 If the diameter AB of a circle be 9, what is the circumference?



Here $7 : 22 :: 9 : 28\frac{2}{7}$; *Or* $\frac{22 \times 9}{7} = 28\frac{2}{7} =$ *circumference required.*

2. If

---

ratio than that of $3\frac{10}{70}$ to 1, and that the perimeter of an inscribed polygon of 96 sides is to the diameter in a greater ratio than that of $3\frac{10}{71}$ to 1, and from thence inferred the ratio abovementioned; as may be seen in his book *de dimensione circuli.* The same proportion was also discovered by *Philo Gedarensis,* and *Apollonius Pergæus,* at a still earlier period, as we are informed by *Eutocius,* in his observations on a work not come to our hands, called *Ocyteboos.*

The proportion of *Vieta* and *Metius* is that of 113 to 355, which is something more exact than the former, *and is the same as the second rule.*

This is a very commodious proportion; for being reduced into decimals, it agrees with the truth as far as the sixth figure inclusively. It was derived from the pretended quadrature of a M. Van-eick, which first gave rise to the discovery.

But the first who ascertained this ratio to any great degree of exactness was *Van Ceulen,* a *Dutchman,* in his book *de Circulo & Adscriptis.* He found that if the diameter of a circle was 1, the circumference would be 3,141592653589793238462643383279502884 *nearly*; which is exactly true to 36 places

2. If the circumference of a circle ADBC be 36 feet, what is the diameter?

$$22 : 7 :: 36$$

```
           36
            7
         ----
     22)252(11 10/22 feet, the diameter.
        22
        ---
         32
         22
         --
         10
```

Or $\frac{36\times7}{22}=\frac{18\times7}{11}=\frac{126}{11}=11\frac{5}{11}$ *as before.*

RULE.

---

places of decimals, and was effected by means of the continual bisection of an arc of a circle, a method so exceedingly troublesome and laborious, that it must have cost him incredible pains. It is said to have been thought so curious a performance, that the numbers were cut on his *tomb-stone*, in *St. Peter's Church-yard*, at *Leyden*. This last number has since been confirmed, and extended to double the number of places, by the late ingenious *Mr. Abraham Sharp*, of *Little Horton*, near *Bradford*, in *Yorkshire*.

But since the invention of *Fluxions*, and the *Summation of Infinite Series*, there have been several methods found out for doing the same thing with much more ease and expedition. The late *Mr. John Machin*, *Professor of Astronomy in Gresham College*, has, by these means, given a quadrature of the circle which is true to 100 places of decimals; and *M. De Lagny*, *M. Euler*, &c. have carried it still farther. All of which proportions are so extremely near the truth, that except the ratio could be completely obtained, we need not wish for a greater degree of accuracy.

The method of obtaining this proportion from the doctrine of fluxions may be shewn as follows:

RULE II.

As 113 is to 355 so is the diameter to the circumference. Or, as 355 is to 113 so is the circumference to the diameter.

EXAMPLES.

1. The diameter of a circle is 9 feet; what is the circumference?

Here, as $113 : 355 :: 9 : 28\frac{31}{113}$.

Or $\frac{355 \times 9}{113} = \frac{3195}{113} = 28.27 =$ *circumference.*

2. If

Let $r =$ radius, $x =$ abciſſa, or verſed ſine, $y =$ ordinate, or ſine, and $z =$ length of the arc.

Then $\overline{2r - x} \times x = 2rx - x^2 = y^2$ by the property of the circle; from which x is found $= r + \sqrt{r^2 - y^2}$, and $\dot{x} = -y\dot{y} \times \overline{r^2 - y^2}|^{-\frac{1}{2}}$.

But $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$ as is ſhewn by the writers on fluxions; whence by ſubſtitution $\dot{z} = r\dot{y} \times \overline{r^2 - y^2}|^{-\frac{1}{2}}$, which thrown into an infinite ſeries gives $\dot{z} = r\dot{y} \times (\frac{1}{r} + \frac{y^2}{2r^2} + \frac{3y^4}{8r^5} + \frac{5y^6}{16r^7} + \frac{35y^8}{128r^9}$ &c.); the fluent which is $z = y \times (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \frac{3.5.7y^8}{2.4.6.8.9r^8}$ &c. $=$ length of the arc in terms of the ſine; which, when y is equal to r, becomes $r \times (1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} + \frac{3.5.7}{2.4.6.8.9}$ &c. $= \frac{1}{4}$ of the circumference.

And if inſtead of y in this ſeries be ſubſtituted its value $\frac{rt}{\sqrt{r^2 + t^2}}$, we ſhall have $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5t^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8}$ &c. $=$ length of the arc in terms of the tangent. Where, by taking the arc ſo that the tangent can be found in terms of the radius, the ſeries will become known, and may be determined to any degree of exactneſs.

Thus,

2. If the diameter of a circle be 10 feet, what is the circumference?

As 113 : 355 :: 10
10

113)3550($31\frac{47}{113}$, *feet the circumference.*
339

160
113

47

Or $\frac{355\times10}{113}=\frac{3550}{113}=31.41=$ *circumference.*

3. What

Thus, if the radius be 1, and the arc be $\frac{1}{8}$ part of the circumference, or 45°, its tangent will be equal to the radius, and the series will become $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$ &c. = arc of 45°, and $8\times(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$ &c.) = whole circumference.

This series is the simplest form that can possibly be obtained, but in order to get another that will converge faster, we must take a smaller arc; as for instance, suppose that of 30°, or $\frac{1}{12}$ part of the circumference.

Then since the tangent of 30°, to radius 1, is $\sqrt{\frac{1}{3}}$, the general series will become $\sqrt{\frac{1}{3}}\times(1-\frac{1}{3\cdot3}+\frac{1}{5\cdot3^2}-\frac{1}{7\cdot3^3}+\frac{1}{9\cdot3^4}$ &c.) = arc of 30°; and $12\sqrt{\frac{1}{3}}\times(1-\frac{1}{3\cdot3}+\frac{1}{5\cdot3^2}-\frac{1}{7\cdot3^3}+\frac{1}{9\cdot3^4}$ &c.) = whole circumference; *and so for any other arc whatever.*

Those

3. What is the diameter of a circle whose circumference is 50?

As 355 : 113 :: 50

```
                 113
                  50
                ----
      355)5650(15 the diameter.
          355
          ----
          2100
          1775
          ----
           325
```

RULE III.

Multiply the diameter by 3.1416 and the product will be the circumference; or

Divide the circumference by 3.1416 and the quotient will be the diameter.

EXAMPLES.

1. If the diameter of a circle be 17 what is the circumference?

```
   3.1416
       17
  -------
   219912
   31416
  -------
  53.4072 = circumference.
```

Those who would wish to see the methods of *Machin*, *Euler*, &c. may consult Dr. Hutton's Mensuration, and a paper of his in the Philosophical Transactions upon this subject.

2. If

2. If the circumference of a circle be 354 what is the diameter?

By Rule I.

As $22 : 7 :: 354 : \frac{354 \times 7}{22} = 112.639 =$ *circumference.*

By Rule II.

As $355 : 113 :: 354 : \frac{354 \times 113}{355} = 112.681 =$ *circumference.*

By Rule III.

As $3.1416 : 1 :: 354 : \frac{354}{3.1416} = 112.681 =$ *circumference, agreeing with the former as far as the third place of decimals.*

3. What is the circumference of a circle whose diameter is 40 feet? *Ans.* 125.6640.

4. If the circumference of a circle be 12, what is the diameter? *Ans.* 3.81972.

5. The earth is a globe, and its circumference is known to be 25000 miles, what is its diameter?

The earth is a globe *Ans.* 7958 *nearly.*

If d be put = diameter of any circle, and c = circumference, the three rules above given, may be expressed thus:

$$\frac{22d}{7} = c; \quad \frac{355d}{113} = c; \quad 3.1416d = c,$$

$$\frac{7c}{22} = d; \quad \frac{113c}{355} = d; \quad \frac{c}{3.1416} = d.$$

And if r, or the radius, be used instead of the diameter, these rules become

$$\frac{44r}{7} = c; \quad \frac{710r}{113} = c; \quad 6.2832r = c,$$

$$\frac{7c}{44} = r; \quad \frac{113c}{710} = r; \quad \frac{c}{6.2832} = r.$$

PRO-

PROBLEM X.

To find the length of any arc of a circle.

RULE I.*

From 8 times the chord of half the arc fubtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc *nearly*.

EXAMPLES.

1. The chord of the whole arc DE is 48, and the verfed fine BC of half the arc is 18: what is the length of the arc DCE?

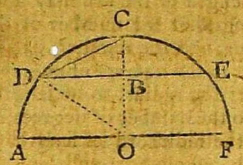
---

* *Demon.* Let the radius OD $= r$, and fine DB $= s$. Then will the chord DC $= \sqrt{s^2 + (r - \sqrt{r^2 - s^2})^2} = s + \frac{s^3}{8r^2} + \frac{7s^5}{128r^4}$ &c. Whence 8 times the chord DC $= 8s + \frac{s^3}{r^2} + \frac{7s^5}{16r^4}$ &c. And 8DC $-$ DE $= 6s + \frac{s^3}{r^2} + \frac{7s^5}{16r^4}$ &c. whofe $\frac{1}{3}$ part is $2s + \frac{s^3}{3r^2} + \frac{7s^5}{48r^4}$ &c.

But the length of the arc DC, whofe fine is s, is known to be $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4}$ &c. and therefore the arc DE $= 2s + \frac{s^3}{3r^2} + \frac{6s^5}{40r^4}$ &c. which differs from $2s + \frac{s^3}{3r^2} + \frac{7s^5}{48r^4}$ &c. only by a fmall quantity, and fhews the rule to be very near the truth. Q. E. D.

Rule 2. If d be put $=$ to the diameter, and $v =$ verfed fine of half the arc, then will $(5d\sqrt{\frac{5d}{5d - 3v}} + 4\sqrt{dv}) \times \frac{2}{5} =$ length of the arc *nearly*.

Here



Here $24^2+18^2=576+324=900$; *and* $\sqrt{900}=30=$ DC. *Whence* $\frac{30\times 8-48}{3}=\frac{240-48}{3}=\frac{192}{3}=64$ *the length of the arc required.*

2. The chord of the whole arc is 50.8, and the chord of half the arc is 30.6: required the length of the arc. *Ans.* 64.6.

3. The length of the whole chord is 6, and the length of the chord of half the arc 3.0538: what is the length of the arc? *Ans.* 6.14768.

RULE II*.

1. Add the ſquare of half the chord to the ſquare of the verſed ſine of half the arc, and this ſum divided by the verſed ſine will give the diameter.

2. Take $\frac{4\frac{1}{5}}{0}$ of the verſed ſine from the diameter, and divide $\frac{2}{3}$ of the verſed ſine by the remainder.

3. To the quotient, laſt found, add 1; and this ſum multiplied by the chord of the whole arc, will give the length of the arc *nearly*.

Note.

* *Demon.* Let the diameter AF $=d$, the verſed ſine CB $=v$, and the chord DE $=c$.

Then

Note. The length of the arc may alſo be found by multiplying together the number of degrees it contains, the radius, and the number .01745329.

Or, as 180 is to the number of degrees in the arc, ſo is 3.1416 times the radius, to its length.

Or, as 3 is to the number of degrees in the arc, ſo is .05236 times the radius, to its length.

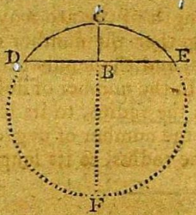
EXAMPLES.

1. If the chord DE be 48, and the verſed ſine CB 10: what is the length of the arc?

Then will the rule be expreſſed by $(\frac{2}{3}v \div \overline{d - \frac{41}{50}v} + 1) \times c$ or by $(\frac{2}{3}v \div \overline{d - \frac{41}{50}v} + 1) \times 2\sqrt{dv - v^2}$, where $d = \frac{c^2}{40} + v$

But $\frac{\frac{2}{3}v}{d - \frac{41}{50}v} + 1 = 1 + \frac{2v}{3d} + \frac{41v^2}{75d^2} + \frac{41 \times 41v^3}{75 \times 50d^3}$ &c. and $2\sqrt{dv - v^2} = 2\sqrt{dv} \times : 1 - \frac{v}{2d} - \frac{v^2}{8d^2}$ &c. Whence $(\frac{2}{3}v \div \overline{d - \frac{41}{50}v} + 1) \times 2\sqrt{dv - v^2} = (1 + \frac{2v}{3d} + \frac{41v^2}{75d^2}$ &c$)$. $\times\ (2\sqrt{dv} \times : 1 - \frac{v}{2d} - \frac{v^2}{8d^2}$ &c.$) = 2\sqrt{dv} \times : 1 + \frac{v}{6d} + \frac{53v^2}{600d^2}$ &c. Now $2\sqrt{dv} \times : 1 + \frac{v}{6d} + \frac{3v^2}{40d^2}$ &c. is known to be the length of an arc whoſe diameter is d, and the verſed ſine of the half thereof v; and this differs from $2\sqrt{dv} \times : + \frac{v}{6d} + \frac{53v^2}{600d^2}$ &c. in the third term only by $\frac{v^2}{75d^2}$, which ſhews the rule to be near an approximation. Q. E. I.

Here



Here $\frac{24^2+18^2}{18}=\frac{576+324}{18}=\frac{900}{18}=50=$ *diameter* CF.

And $18\times\frac{2}{3}\div(50-\frac{41}{50}\times 18)=12\div\overline{50-14.76}=12\div 35.24=.3405$.

Whence $(1+.3405)\times 48=1.3405\times 48=64.3440=$ *length of the arc required.*

2. The chord of the whole arc is 7, and the versed sine 2: what is the length of the arc?

Ans. 8.8439.

3. The chord of the whole arc is 40, and the versed sine 15: what is the length of the arc?

Ans. 53.62.

PROBLEM XI.

To find the area of a circle.

RULE I*.

Multiply half the circumference by half the diameter, and the product will be the area.

Or take $\frac{1}{4}$ of the product of the whole circumference and diameter.

EXAM-

---

* *Demon.* A circle may be considered as a regular polygon of an infinite number of sides, the circumference being equal to

EXAMPLES.

1. What is the area of a circle whose diameter is 42, and circumference 131.946?

$$2)131.946$$

$$65.973 = \tfrac{1}{2} \text{ circumference.}$$
$$21 = \tfrac{1}{2} \text{ diameter.}$$

65973
131946

$1385.433 =$ *area required.*

2. What is the area of a circle whose diameter is 10 feet 6 inches, and circumference 31 feet 6 inches?

fe.	*in.*		
15	9 =	15.75 =	$\tfrac{1}{2}$ *circumference,*
5	3 =	5.25 =	$\tfrac{1}{2}$ *diameter.*

7875
3150
7875

82.6875
12

8.2500

Anf. 82 *feet,* 8 *inches.*

to the perimeter, and the radius to the perpendicular. But the area of a regular polygon is equal to half the perimeter multiplied by the perpendicular, and confequently the area of a circle is equal to half the circumference multiplied by the radius, or half the diameter. Q. E. D.

This rule may be otherwife demonftrated by the doctrine of fluxions.

3. What

3. What is the area of a circle whose diameter is 1, and circumference 3.14159? *Ans.* .7854.

4. What is the area of a circle whose diameter is 7 and circumference 22? *Ans.* $38\frac{1}{2}$.

RULE II.*

Multiply the square of the diameter by .7854, and the product will be the area.

EXAM-

---

* *Demon.* All circles are to each other as the squares of their diameters. (Euc. XII. 2.)

But the area of a circle whose diameter is 1, is .7854 &c. (by Rule I.) Therefore $1^2 : d^2 :: .7854 \text{ \&c.} : \frac{.7854 \text{ \&c.} \times d^2}{1}$ $= .7854 \text{ \&c.} \times d^2 =$ area of a circle whose diameter is d. Q. E. D.

The following proportions are those of *Metius* and *Archimedes*.

As 452 : 355 :: square of the diameter : Area.
As 14 : 11 :: square of the diameter : Area.

If the circumference be given, instead of the diameter, the area may be found as follows:

The square of the circumference × .07958 = Area.
As 88 : 7 :: square of the circumference : Area.
As 1420 : 113 :: square of the circumference : Area.

And if d be the diameter, c the circumference, a the area, and $p = 3.14159$ &c. then:

$$1.\ d = \frac{c}{p} = \frac{4a}{c} = 2\sqrt{\frac{a}{p}}$$

$$2.\ c = pd = \frac{4a}{d} = 2\sqrt{pa}$$

$$3.\ a = \frac{pd}{4} = \frac{c}{4p} = \frac{dc}{4}$$

The

EXAMPLES.

1. What is the area of a circle whose diameter is 5?

$$\begin{array}{r} .7854 \\ 25 = \textit{square of the diameter.} \\ \hline 39270 \\ 15708 \\ \hline 19.6350 = \textit{the answer.} \end{array}$$

2. What is the area of a circle whose diameter is 7? *Ans.* 38.4846.

3. What is the area of a circle whose diameter is 4.5? *Ans.* 15.90435.

4. How many square yards are there in a circle whose diameter is $3\frac{1}{2}$ feet? *Ans.* 1.069016.

PROBLEM XII.

To find the area of a sector, or that part of a circle which is bounded by any two radii and their included arc.

The following table will also shew most of the useful problems, relating to the circle and its equal or inscribed square.

1. diameter × .8862 = side of an equal square.
2. circumf. × .2821 = side of an equal square.
3. diameter × .7071 = side of the inscribed square.
4. circumf. × .2251 = side of the inscribed square.
5. area × .6366 = side of the inscribed square.
6. side of a square × 1.4142 = diameter of its circumf. circle.
7. side of a square × 4.443 = circumf. of its circumf. circle.
8. side of a square × 1.128 = diameter of an equal circle.
9. side of a square × 3.545 = circumf. of an equal circle.

RULE.

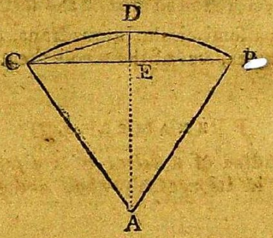
RULE.*

Multiply the radius, or half the diameter, by half the length of the arc of the sector, and the product will be the area.

Note. The length of the arc may be found by either of the last problems.

EXAMPLES.

1. The radius A B is 40, the chord B C of the whole arc 50, and the chord C D of half the arc 27: required the area of the sector.



Here $\frac{27\times8-50}{3}=\frac{216-50}{3}=\frac{166}{3}=55.\dot{3}=$ *length of the arc* C D B.

And $\frac{55.\dot{3}}{2}\times40=27.\dot{6}\times40=1109.\dot{6}=$ *area of the sector required.*

* The rule for finding the area of the sector, is, evidently, the same as that for finding the area of the whole circle.

 2. Re-

2. Required the area of a sector of a circle, whose radius is 25, and the length of the arc 21.5.

Ans. 268.75.

3. What is the area of a sector whose radius is 10, the chord of the whole arc 20, and the chord of half the arc 11?

Ans. 113.3.

4. Find the area of the sector whose radius is 9, and the chord of whose arc is 6.

Ans. 27.5267835.

RULE II.*

As 360 is to the degrees in the arc of a sector, so is the area of the whole circle, whose radius is equal to that of the sector, to the area of the sector required.

Note. For a semi-circle, a quadrant, &c. take one half, one quarter, &c. of the whole area.

EXAMPLES.

1. The radius of a sector of a circle is 20, and the degrees in its arc 22; what is the area of the sector?

---

* *Demon.* Let $r =$ radius, $d =$ number of degrees in the arc of the sector, and $A =$ its area.

Then will $4r^2 \times .7854 = r^2 \times 3.1416 =$ area of the whole circle, and $2r \times 3.1416 =$ its circumference.

Also $360 : 2r \times 3.1416 :: d : \frac{2dr \times 3.1416}{360} =$ length of the arc of the sector. But $\frac{2dr \times 3.1416}{360} \times \frac{1}{2} \times r = \frac{dr^2 \times 3.1416}{360} = A$, by the last rule. And consequently $360 : d :: r^2 \times 3.1416 : A$.

Q. E. D.

.7854
1600 = *square of the diameter.*

4712400
7854

360 : 22 :: 1256.6400 = *area of the whole circle.*
22

251328
251328

36.0)2764.608 (7.6 = *area of the sector.*
252

244
216

286
&c.

2. Required the area of a sector whose radius is 25, and the length of its arc 147 degrees 29 minutes.

Ans. 804.4017.

3. Required the area of a semicircle whose radius is 13.

Ans. 265.4652.

4. Required the area of a quadrant whose radius is 21.

Ans. 346.3614.

PROBLEM XIII.

To find the area of a segment of a circle.

RULE I.

1. Find the area of the sector, having the same arc with the segment, by one of the last problems.

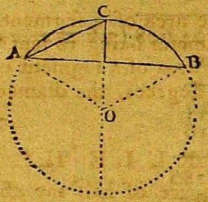
 2. Find

2. Find the area of the triangle formed by the chord of the segment, and the radii of the sector.

3. Then the sum, or difference, of these areas, according as the segment is greater or less than a semicircle, will be the area required.

EXAMPLES.

1. The radius OB is 10, the chord AB is 18, and the chord AC 10; what is the area of the segment ABC?



Here $\frac{10\times8-18}{3}=\frac{80-18}{3}=\frac{62}{3}=20.\dot{6}=$ *arc* ACB.

And $\frac{20.6}{2}\times10=10.\dot{3}\times10=103.\dot{3}=$ *area of the sector* OACB.

Also $\frac{18+10+10}{2}=\frac{38}{2}=19=\frac{1}{2}$ *sum of the sides* AO, OB, BA.

Whence $19-10=9=$ 1st *and* 2d *diff. and* $19-18=1=$ 3d *diff.*

And $\sqrt{19\times9\times9\times1}=\sqrt{1539}=39.23=$ *area of the* $\triangle$ BOA.

And $103.\dot{3}-39.23=64.10=$ *area of the segment required.*

It being in this case less than a semicircle.

2. Re-

2. Required the area of a ſegment whoſe height is 2, and chord 20. *Anſ.* 26.878.

3. Required the area of a ſegment of a circle whoſe radius is 24, the chord of the whole arc 20, and the chord of half the arc 10.2. *Anſ.* 28.07.

4. Required the area of a ſegment of a circle whoſe chord is 12, and the radius of the circle 10.
Anſ. 16.3274.

5. Required the area of a ſegment of a circle whoſe chord is 16, and the diameter of the circle 20.
Anſ. 44.7292.

6. What is the area of a ſegment, whoſe arc is a quadrant, the diameter being 18 feet? *Anſ.* 63.1174.

7. What is the area of a ſegment of a circle whoſe arc contains 280 degrees, the diameter being 50?
Anſ. 1834.9191.

RULE II.*

1. Add the ſquare of half the chord of the ſegment to the ſquare of its height, and multiply the ſquare root of the ſum by 4.

2. To

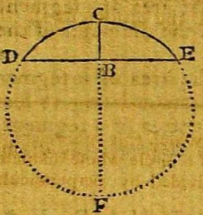
---

* *Demon.* Let $c =$ BE, $v =$ CB, and $d =$ diameter CF; then will the rule be expreſſed by $(2c + \frac{4}{3}\sqrt{c^2 + v^2}) \times \frac{2}{5}v$. But $(2c + \frac{4}{3}\sqrt{c^2 + v^2}) \times \frac{2}{5}v = (\sqrt{dv - v^2} + \frac{2}{3}\sqrt{dv}) \times \frac{4}{5}v$ $= \frac{4}{5}v \times \sqrt{dv} + \frac{4}{5}v\sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{8d^2} - \frac{3v^3}{48d^3}$ &c.$) = \frac{4}{3}v$ $\sqrt{dv} - \frac{2v^2\sqrt{dv}}{5d} - \frac{2v^3\sqrt{dv}}{20d^2}$ &c. $= 2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d}$ $- \frac{v^2}{20d^2} - \frac{v^3}{72d^3}$ &c.) And ſince BE $= \sqrt{dv - v^2}$, and CB $= v$, therefore $\dot{v}\sqrt{dv} \times : 1 - \frac{v}{2d} - \frac{v^2}{8d^2} - \frac{3v^3}{48d^3}$ &c. will be the fluxion of the ſegment; and its fluent, $v\sqrt{dv}$

2. To $\frac{1}{3}$ of the number laſt found, add the whole chord of the ſegment, and this ſum multiplied by $\frac{2}{5}$ of the height will give the area required.

EXAMPLES.

1. The chord DE of the ſegment DEC is 40, and its height, or verſed ſine, BC is 10: what is the area of the ſegment?



Here $4\sqrt{20^2+10^2}=4\sqrt{400+100}=4\sqrt{500}=4\times 22.36=89.44$.

And $(\frac{89.44}{3}+40)\times\frac{2\times 10}{5}=\overline{29.81+40}\times 4=69.81\times 4=279.24=$ *area required.*

$\times : \frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$ &c. will be the value of half the ſegment. Conſequently its double, $2v\sqrt{dv}\times(\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}-\frac{v^3}{72d^3}$ &c.) will be the value of the whole ſegment; which differs from the expreſſion for the rule only in the third term, and therefore ſhews the *approximation* required. Q. E. D.

This rule was firſt given by *Sir Iſaac Newton*, and publiſhed by *Jones*, *Wallis*, and *Colſon*.

If $c=$ chord of the ſegment, and $v=$ its height; then $(7\sqrt{\frac{1}{4}c^2+\frac{2}{7}v^2}+\frac{4}{3}\sqrt{\frac{1}{4}c^2+v^2})\times\frac{4}{25}v=$ area *very nearly*, which is another approximating rule, ſufficiently eaſy in its application.

2. The chord is 20, and the height, or versed sine 2; required the area of the segment.

Ans. 26.85908.

3. The length of the chord is 48, and the height of the segment 18; what is the area? *Ans.* 633.6.

RULE III.*

1. Divide the height, or versed sine, by the diameter, and find the quotient in the table of versed sines.

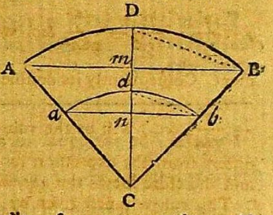
2. Mul-

---

* The table to which this rule refers, is formed of the areas of the segments of a circle whose diameter is 1; and which is supposed to be divided by perpendicular chords into 1000 equal parts.

The reason of the rule itself depends upon this property—That the versed sines of similar segments are as the diameters of the circles to which they belong, and the areas of those segments as the squares of the diameters; which may be thus demonstrated.

Let ADBA and $adba$ be any two similar segments, cut off from the similar sectors ADBCA and $adbca$, by the chords AB and ab; and let the perpendicular CD bisect them.



Then, by similar triangles, $DB : db :: BC : bc$, and $DB : db :: Dm : dn$; whence, by equality, $BC : bc :: Dm : dn$, or $2BC : 2bc :: Dm : dn$.

Again, since similar segments are as the squares of their chords, it will be $AB^2 : ab^2 ::$ seg. ADBA : seg. $adba$; but $AB^2 : ab^2 :: CB^2 : cb^2$, whence, by equality, seg. ADBA : seg. $adba :: CB^2 : cb^2$, or seg. ADBA : seg. $adba :: 4CB^2 : 4cb^2$. Q. E. D.

 Now,

2. Multiply the number on the right hand of the versed sine, by the square of the diameter, and the product will be the area.

EXAMPLES.

1. If the chord of a circular segment be 40, its versed sine 10, and the diameter of the circle 50: what is the area?

$$
\begin{array}{rl}
5.0)1.0 & \\
\hline
.2 & = \textit{tabular versed sine.} \\
.111823 & = \textit{tabular segment.} \\
2500 & = \textit{square of } 50. \\
\hline
55911500 & \\
223646 & \\
\hline
279.557500 & = \textit{area required.}
\end{array}
$$

2. The cord of the segment is 20, the versed sine 5, and the diameter 25: what is the area?

Ans. 69.889375.

PROBLEM XIV.

To find the area of a circular zone, or the space included between any two parallel chords and their intercepted arcs.

Now, if d be put equal to any diameter, and v the versed sine, it will be $d : v :: 1$ (diameter in the table) : $\frac{v}{d}$ = versed sine of a similar segment in the table, whose area let be called a.

Then $1^2 : d^2 :: a : ad^2$ = area of the segment whose height is v, and diameter d, as in the rule.

RULE.

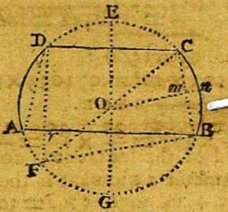
RULE.*

1. Find the area of that part of the zone which forms a trapezoid by problem 6, and the area of the ſegment B *n* C B by problem 13, rule 3.

2. Add the area of the trapezoid to twice the area of the ſegment, and it will give the area of the zone required.

EXAMPLES.

1. The greater chord A B is 20, the leſs D C 15, and their diſtance D *r* $17\frac{1}{2}$: required the area of the zone A B C D.



---

* The reaſon of this rule is too obvious to require a demonſtration.

Note. When the two parallel ſides of the zone are equal, the chord of the ſmall ſegment will be equal to the breadth of the zone, and its height will be equal to $\sqrt{\frac{1}{4}AB^2 + \frac{1}{4}as^2} - \frac{1}{2}AB$.

And when one of the ſides is the diameter of the circle, the chord of the ſame ſegment will be $\sqrt{as^2 + D^2}$, and its height $= \frac{1}{2}AB - \sqrt{\frac{1}{4}AB^2 - \frac{1}{4}as^2 - \frac{1}{4}D^2}$ where $D = AB - DC$.

Another method, is by finding the areas of each of the ſegments A B E A, D C E D, the difference of which is the area of the zone required; but this rule when the ſegments are large, and any of the approximating theorems are uſed, is not ſo accurate as the former.

 Here

Here $\sqrt{Dr^2 + rA^2} = \sqrt{17.5^2 + 2.5^2} = \sqrt{306.25 + 6.25}$ $= \sqrt{312.5} = 17.6776 =$ AD *or* CB.

And Dr $(17\frac{1}{2})$: *r*A $(2\frac{1}{2})$ :: B*r* $(17\frac{1}{2})$: $2\frac{1}{2} =$ *r*F; *or* $2\frac{1}{2} + 17\frac{1}{2} = 20 =$ DF.

Also $\sqrt{DF^2 + DC^2} = \sqrt{20^2 + 15^2} = \sqrt{400 + 225}$ $= \sqrt{625} = 25 =$ CF; *or* $\frac{25}{2} = 12\frac{1}{2} =$ oC *or* o*n*.

*And r*D $(17\frac{1}{2})$: AD (17.6776) :: *r*B $(17\frac{1}{2})$: 17.6776 $=$ FB; *or* $\frac{17.6776}{2} = 8.8388 =$ o*m*; *and* $12.5 - 8.8388$ $= 3.6612 =$ *mn* $=$ *height of the segment.*

Therefore $\frac{3.6612}{25} = .146448 =$ *tab. versed sine*; *and* $.071033 =$ *tab. segment.*

Whence $.071033 \times 25^2 = .071033 \times 625 = 44.395625$ $=$ *area of the segment* B*n*CB.

Also $\frac{15 + 20}{2} \times 17\frac{1}{2} = \frac{35}{2} \times \frac{35}{2} = \frac{1225}{4} = 306.25 =$ *area of the trapezoid* ABCD.

And $306.25 + 44.395625 \times 2 = 306.25 + 88.79025$ $= 394.04025 =$ *area of the zone required.*

2. The greater side is 96, the lesser 60, and the breadth $17\frac{1}{2}$: what is the area of the zone?

Ans. 2136.7712.

3. One end of a circular zone is 48, the other end is 30, and the breadth is 13; what is the area of the zone? *

Ans. 534.4249.

PROBLEM XV.

To find the area of a circular ring, or the space included between the circumferences of two concentric circles.

---

* In any of these questions, when the radius or diameter of the circle is given, the operation is much more easily performed.

RULE.

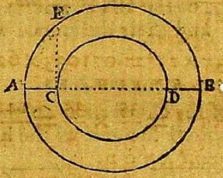
RULE.*

The difference between the areas of the two circles will be the area of the ring.

Or, Multiply the sum of the diameters by their difference, and this product again by .7854, and it will give the area required.

EXAMPLES.

1. The diameters A B and C D are 20 and 15: required the area of the circular ring, or the space included between the circumferences of those circles.



Here $\overline{AB + CD} \times \overline{AB - CD} = 35 \times 5 = 175$; *and* $175 \times .7854 = 137.4450$ = *area of the ring required.*

---

* *Demon.* The area of the circle A E B A $= AB^2 \times .7854$, and the area of the small circle C D is $= CD^2 \times .7854$; therefore the area of the ring $= AB^2 \times .7854 - CD^2 \times .7854 = \overline{AB + CD} \times \overline{AB - CD} \times .7854$. Q. E. D.

Coroll. If C E be a perpendicular at the point C, the area of the ring will be equal to that of a circle whose radius is C E.

Rule 2. Multiply half the sum of the circumferences by half the difference of the diameters, and the product will be the area.

This rule will also serve for any part of the ring, using half the sum of the intercepted arcs for half the sum of the circumferences.

 2. The

2. The diameters of two concentric circles are 16 and 10: what is the area of the ring formed by thoſe circles? *Anſ.* 122.5224.

3. The two diameters are 21.75 and 9.5: required the area of the circular ring. *Anſ.* 300.6609.

4. Required the area of the ring, the diameters of whoſe bounding circles are 6 and 4. *Anſ.* 15.708.

PROBLEM XVI.

To find the areas of lunes, or the ſpaces included between the interſecting arcs of two eccentric circles.

RULE.*

Find the areas of the two ſegments from which the lune is formed, and their difference will be the area required.

EXAMPLES.

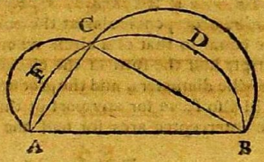
The length of the chord AB is 40, the height DC 10, and DE 4: required the area of the lune ACBEA.

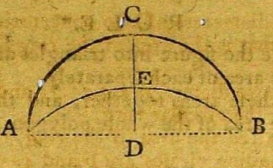
---

* Whoever wiſhes to be acquainted with the properties of lunes, and the various theorems ariſing from them, may conſult *Mr. Whiſton's Commentary on Tacquet's Euclid,* where they will find this ſubject very ingeniouſly managed.

The following property is one of the moſt curious.

If ABC be a right angled triangle, and ſemicircles be deſcribed on the three ſides as diameters, then will the ſaid triangle be equal to the two lunes D and F taken together.





Here $4\sqrt{\overline{AD^2+DC^2}} = 4\sqrt{\overline{400+100}} = \overline{22.36} \times 4 = 89.44$.

And $(\frac{89.44}{3}+40) \times \frac{2 \times 10}{5} = \overline{29.81+40} \times 4 = 279.24$ = *area of the segment* ABCA.

Also $4\sqrt{\overline{AD^2+DE^2}} = 4\sqrt{\overline{400+16}} = 20.396 \times 4 = 81.854$.

And $(\frac{81.584}{3}+40) \times \frac{2 \times 4}{5} = \overline{27.194+40} \times 1.6 = 107.5104$ = *area of the segment* ABEA.

Whence $279.24 - 107.5104 = 171.7296$ = *area of the lune required**.

2. The chord is 20, and the heights of the segments 10 and 2: required the area of the lune.

Ans. 128.555.

3. The length of the chord is 48, and the heights of the segments 18 and 7: what is the area of the lune? *Ans.* 405.8676.

PROBLEM XVII.

To find the area of an irregular polygon, or a figure of any number of sides.

* *Note.* The areas of the segments in this and the following examples may be more accurately found by Rule 3, page 77.

 RULE.

RULE.*

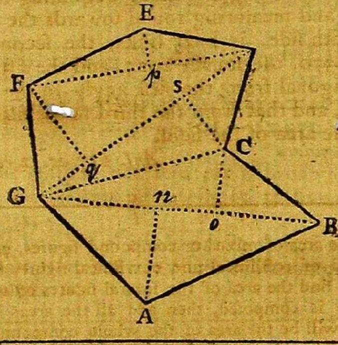
1. Divide the figure into triangles and trapeziums, and find the area of each separately.

2. Add these areas together, and the sum will be equal to the area of the whole polygon.

EXAMPLES.

1. Required the area of the irregular figure A B C D E F G A, the following lines being given.

GB = 30.5,	An = 11.2,	Co = 6
GC = 29	Fq = 11	Cs = 6.6
FD = 24.1	Ep = 4	......



---

* When any part of the figure is bounded by a curve, the area may be found as follows:

Rule. 1. Erect any number of perpendiculars upon the base, at equal distances, and find their lengths.

2. Add the lengths of the perpendiculars, thus found, together, and the sum divided by their number will give the *mean breadth*.

3. Multiply the mean breadth by the length of the base, and it will give the area of that part of the figure required.

Here

Here $\frac{\text{A}n+\text{C}o}{2}+\text{GB}=\frac{11.2+6}{2}\times 30.5=8.6\times 30.5$ $=262.3=$ *area of the trapezium* ABCG.

And $\frac{\text{F}q+\text{C}s}{2}\times\text{GD}=\frac{11+6.6}{2}\times 2.9=8.8\times 2.9=$ $255.2=$ *area of the trapezium* GCDF.

Also $\frac{\text{FD}\times\text{E}p}{2}=\frac{24.8\times 4}{2}=\frac{99.2}{2}=49.6=$ *area of the triangle* FDE.

Whence $262.3+255.2+49.6=567.1=$ *area of the whole figure required.*

2. * In a pentangular field, beginning with the ſouth ſide, and meaſuring round towards the eaſt, the firſt, or ſouth ſide, is 2735 links, the ſecond 3115, the third 2370, the fourth 2925, and the fifth 2220; alſo the diagonal from the firſt angle to the third is 3800 links, and that from the third to the fifth 4010: required the area of the field.

Anſ. 117 *ac.* 2 *ro.* 28 *po.*

To find the area of mixed or compound figures, or ſuch as are compoſed of rectilineal and curvilineal figures together: the rule is to find the area of the ſeveral figures of which the whole figure is compoſed, then add all the areas together, and the ſum will be the area of the whole compound figure. And in the ſame manner may any irregular field or piece of land be meaſured, by dividing it into trapeziums and triangles, and finding the area of each ſeparately.

* *Note.* As this figure conſiſts of three triangles, all of whoſe ſides are given, by calculating their areas according to problem 3, the ſum will be the area of the whole figure accurately, without drawing perpendiculars from the angles to the diagonals.

The ſame thing may alſo be done in moſt other caſes of this kind.

OF

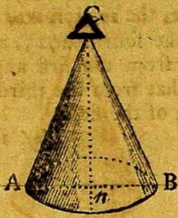
OF THE

CONIC SECTIONS.

DEFINITIONS.

1. THE *conic sections* are such plain figures as are formed by the cutting of a cone.

2. * A *cone* is a solid described by the revolution of a right-angled triangle about one of its legs, which remains fixed.



3. The *axis of a cone* is the right line about which the triangle revolves.

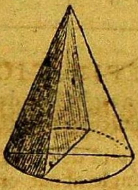
* This is Euclid's definition of a cone, and is that which is generally best understood by a learner; but the following one is more general.

Conceive the right line C B to move upon the fixed point C as a centre, and so as continually to touch the circumference of the circle A B, placed in any position, except in that of a plane which passes through the said point; and then that part of the line which is intercepted between the fixed point and the periphery of the circle will generate the convex superficies of a cone.

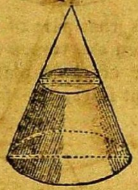
4. The

4. The *base of a cone* is the circle which is described by the revolving leg of the triangle.

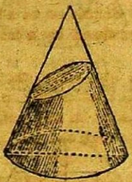
5. If a cone be cut through the vertex, by a plane perpendicular to that of the base, the section will be a *triangle.*



6. If a cone be cut into two parts, by a plane parallel to the base, the section will be a *circle.*



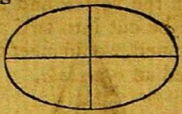
7. If a cone be cut by a plane which passes through its two slant sides in an oblique direction, the section will be an *ellipsis.*



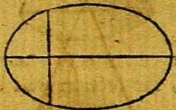
8. If two lines be drawn through the centre of an ellipsis, perpendicular to each other, and terminated both

both ways by the circumference, they are called the *tranſverſe* and *conjugate diameters*, or *axes*.

The longeſt line is the tranſverſe axis, and the ſhorteſt the conjugate.

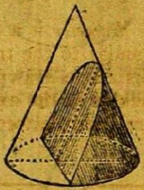


9. *An ordinate* is a right line drawn from any point of the curve, perpendicular to either of the diameters.

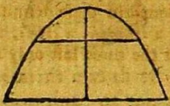


10. *An abſciſſa* is that part of the diameter which is contained between the vertex and the ordinate.

11. If a cone be cut by a plane, which is parallel to either of its ſlant ſides, the ſection will be *a parabola*.



12. *The axis of a parabola* is a right line drawn from the vertex, ſo as to divide the figure into two equal parts.



13. The

13. *The ordinate* is a right line drawn from any point in the curve perpendicular to the axis.

14. *The abſciſſa* is that part of the axis which is contained between the vertex and the ordinate.

15. * If a cone be cut into two parts, by a plane, which being continued, would meet the oppoſite cone, the ſection is called an *hyperbola*.

16. *The tranſverſe diameter* or axis of an hyperbola is that part of the axis intercepted between the two oppoſite cones.

17. *The conjugate diameter* is a line drawn through the centre perpendicular to the tranſverſe.

18. *An ordinate* is a line drawn from any point in the curve perpendicular to the axis; and the *abſciſſa* is the diſtance intercepted between that ordinate and the vertex.

---

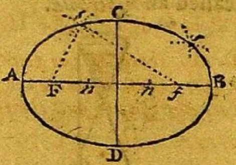
* The two oppoſite cones, in this definition, are ſuppoſed to be generated together, by the revolution of the ſame line.

All the figures which can poſſibly be formed by the cutting of a cone, are mentioned in theſe definitions, and are the five following ones; *viz.* a *triangle*, a *circle*, an *ellipſis*, a *parabola*, and an *hyperbola*; but the three laſt only are uſually called the *conic ſections*.

PRO-

PROBLEM I.

To describe an ellipsis, the transverse and conjugate diameters being given.



*Construction.** 1. Draw the transverse and conjugate diameters, A B, C D, bisecting each other perpendicularly in the centre *o*.

2. With the radius A *o*, and centre C, describe an arc, cutting A B in F *f*; and these two points will be the foci of the ellipse.

3. Take any number of points *n n*, &c. in the transverse diameter A B, and with the radii A *n*, *n* B,

---

* It is a known property of the ellipse, that the sum of two lines drawn from the foci, to meet in any point in the curve, is equal to the transverse diameter, and from this the truth of the construction is evident.

From the same principle is also derived the following method of describing an ellipse, by means of a string and two pins.

Having found the foci F, *f*, as before, take a thread of the length of the transverse diameter, and fasten its ends with two pins in the points F, *f*; then stretch the thread F *s f* to its greatest extent, and it will reach to the point *s* in the curve; and by moving a pencil round within the thread, keeping it always stretched, it will trace out the curve required.

and

and centres F *f* describe arcs interfecting each other in *s*, *s*, &c.

4. Through the points *s*, *s*, &c. draw the curve A *s* C B D, and it will be the circumference of the ellipfe required.

PROBLEM II.

In an ellipfis, any three of the four following terms being given, viz. the tranfverfe and conjugate diameters, an ordinate and its abfciffa, to find the fourth.

CASE I.

When the tranfverfe, conjugate, and abfciffa are known, to find the ordinate.

RULE.*

As the tranfverfe diameter is to the conjugate,
So is the fquare root of the rectangle of the two abfciffas,
To the ordinate which divides them.

EXAMPLES.

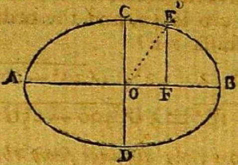
1. In the ellipfis A D B C, the tranfverfe diameter A B is 120, the conjugate diameter C D is 40, and

---

* Let $t =$ the tranfverfe diameter, $c =$ conjugate, $x =$ any abfciffa, and $y =$ ordinate. Then will the general equation, expreffing the property of the ellipfe, be $t^2 : c^2 :: x \times (t-x) : y^2$; and from this the four rules here given are deduced, the one above being $y = \frac{c}{t}\sqrt{x \times \overline{t-x}}$.

the

the abſciſſa B F 24: what is the length of the ordinate E F?



Here 120 (A B) : 40 (C D) :: $\sqrt{96 \times 24}$ (A F × F B) : $\frac{40}{120}\sqrt{96 \times 24} = \frac{1}{3}\sqrt{2304} = \frac{1}{3} \times 48 = 16 =$ E F *the ordinate required.*

2. If the tranſverſe diameter be 35, the conjugate 25, and the abſciſſa 28: what is the ordinate?

Anſ. 10.

CASE II.

When the tranſverſe, conjugate, and ordinate are known, to find the abſciſſa.

RULE.*

As the conjugate diameter is to the tranſverſe,

So is the ſquare root of the difference of the ſquares of the ordinate and ſemi-conjugate,

To the diſtance between the ordinate and centre.

And this diſtance being added to and ſubtracted from the ſemi-tranſverſe, will give the two abſciſſas required.

---

* This rule in algebraic terms is as follows: The greater abſciſſa $x = t + \frac{t}{c}\sqrt{\frac{1}{4}c^2 - y^2}$, or the leſs $x = t - \frac{t}{c}\sqrt{\frac{1}{4}c^2 - y^2}$.

EXAM-

EXAMPLES.

1. The transverse diameter AB is 120, the conjugate diameter CD is 40, and the ordinate FE is 16: what is the abscissa FB?

Here 40 (CD) : 120 (AB) :: $\sqrt{20^2-16^2}$ ($\sqrt{OB^2-FE^2}$) : $\frac{120}{40}\sqrt{20^2-16^2}=3\sqrt{400-256}=3\sqrt{144}=3\times 12=36=$ OF, *the distance from the centre.*

Whence 60 (OB) − 36 (OF) = 24 = BF
And 60 (OA) + 36 (OF) = 96 = AF } = *two abscissas required.*

2. What are the two abscissas to the ordinate 10, the diameters being 35 and 23?

Ans. 7 *and* 28.

CASE III.

When the conjugate, ordinate, and abscissa are known, to find the transverse.

RULE.*

1. To, or from, the semi-conjugate, according as the less or greater abscissa is used, add or subtract the square root of the difference of the squares of the ordinate and semi-conjugate.

2. Then, as the square of the ordinate, is to the rectangle of the conjugate and abscissa, so is the sum or difference above found, to the transverse diameter required.

---

* This rule in algebraic terms is as follows: $t=(c+2\sqrt{\frac{1}{4}c^2-y^2})\times\frac{cx}{2y^2}$, or $t=(c-2\sqrt{\frac{1}{4}c^2-y^2})+\frac{cx}{2y^2}$ according as the greater or less abscissa is used.

EXAM-

EXAMPLES.

1. The conjugate diameter C D is 40, the ordinate E F is 16, and the abſciſſa F B 24: required the tranſverſe A B.

Here 20 (O B) $+ \sqrt{\overline{20^2 - 16^2}}$ $(\sqrt{\overline{OB^2 - EF^2}}) = 20 + \sqrt{400 - 256} = 20 + \sqrt{144} = 20 + 12 = 32.$

And 16^2 $(EF^2) : 40 \times 24$ $(CD \times BF) :: 32 : \frac{40 \times 24 \times 32}{16^2} = \frac{40 \times 24 \times 2}{16} = \frac{5 \times 24 \times 2}{2} = 5 \times 24 =$ 120 *the tranſverſe diameter required.*

2. If an ordinate and its leſſer abſciſſa be 10 and 7, and the conjugate 25: what is the tranſverſe?

Anſ. 35.

CASE IV.

The tranſverſe, ordinate, and abſciſſa being given, to find the conjugate.

RULE.*

As the ſquare root of the product of the two abſciſſas, is to the ordinate,

So is the tranſverſe diameter to the conjugate.

EXAMPLES.

1. The tranſverſe A B is 120, the ordinate E F 16, and the abſciſſa F B 24: required the conjugate.

---

* The rule in algebraic terms is $ty \times \frac{1}{\sqrt{t-x \cdot x}} = c$, the conjugate, or ſhorteſt diameter.

Here

Here $\sqrt{24 \times 96}$ ($\sqrt{BF \times AF}$) : 16 (FF) :: 120 (AB) : $16 \times 120 \div \sqrt{24 \times 96} = 16 \times 120 \div \sqrt{2304} = 16 \times 120 \div 48 = \frac{15 \times 120}{48} = \frac{120}{3} = 40$ *the conjugate diameter required.*

2. The tranſverſe diameter is 35, the ordinate is 10, and its abſciſſa 7: what is the conjugate?

Anſ. 25.

PROBLEM III.

To find the circumference of an ellipſe, the tranſverſe and conjugate diameters being known.

RULE*.

Multiply the ſquare root of half the ſum of the ſquares of the two diameters by 3.1416, and the product will be the circumference *nearly*.

EXAM-

---

* *Demon.* Let t = tranſverſe diameter, c = conjugate, p = 3.1416, and $d = 1 - \frac{c^2}{t^2}$. Then will $pt \times (1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c.) = circumference of the ellipſe, as is ſhewn by the writers on fluxions.

Now the rule given above is $p\sqrt{\frac{t^2+c^2}{2}} = pt\sqrt{(\frac{1}{2} + \frac{c^2}{2t^2})} = pt\sqrt{\{1 - \frac{1}{2} + \frac{c^2}{2t^2}\}}$ = (by ſubſtitution) $pt\sqrt{(1 - \frac{d}{2})} = pt \times : 1 - \frac{d}{2.2} - \frac{d^2}{2^3.4} - \frac{3d^3}{2^4.4.6}$ &c. But the three firſt terms

EXAMPLES.

1. The transverse diameter is 24 and the conjugate 20: required the circumference of the ellipse.

Here $\sqrt{\frac{AB^2 - CD^2}{2}} = \sqrt{\frac{24^2 + 20^2}{2}} = \sqrt{\frac{576 + 400}{2}}$ $= \sqrt{288 + 200} = 488 = 22.09$.

And 22.09 × 3.1416 = 69.397944 *the circumference required.*

2. The axes are 24 and 18: what is the circumference? *Ans.* 66.6433.

---

terms of this series differ from the three first terms of the former only by $\frac{d^2}{64}$; therefore the rule is shewn to be an approximation. Q. E. D.

Rule 2. Multiply ½ the sum of the two diameters by 3.1416, and the product will give the circumference *exact enough to answer most practical purposes.*

Rule 3. Find the circumference both from the last rule and that given above, and ½ the sum of the results will give the circumference *extremely near.*

Note. If a = semi-transverse B O, c = semi-conjugate C O, and x = distance O F, of the ordinate E F from the centre, then will the arc C E be $= x \times (1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4$ $+ \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c.

The following may serve as a practical rule for finding the length of the arc C E.

Find the length of a circular arc intercepted by O E and O C, and whose radius is ½ the sum of O E and O C, and it will be the elliptic arc C E *nearly.*

PRO-

PROBLEM IV.

To find the area of an ellipſe, the tranſverſe and conjugate diameters being given.

RULE.*

Multiply the tranſverſe diameter by the conjugate, and the product again by .7854, and the reſult will be the area.

Or multiply .7854 by one axe, and the product by the other.

EXAMPLES.

1. What is the area of an ellipſe whoſe tranſverſe diameter is 24, and the conjugate 18?

Here 24 × 18 × .7854 = 339.2928 = *area required.*

Or,

```
 .7854
    24=tranſverſe.
 ------
 31416
15708
 ------
18.8496
    18=conjugate.
 ------
1507968
188496
 ------
339.2928=area before.
```

2. If the axes of an ellipſe be 35 and 25, what is the area? *Anſ.* 687.225.

3. Required the area of an ellipſis whoſe two axes are 70 and 50. *Anſ.* 2748.9.

---

* The demonſtration of this rule is contained in that of the next problem.

PROBLEM V.

To find the area of an elliptic ſegment, whoſe baſe is parallel to either of the axes of the ellipſe.

RULE.*

1. Divide the height of the ſegment by that axe of the ellipſe of which it is a part, and find, in the table, a circular ſegment whoſe verſed ſine is equal to this quotient.

2. Multiply the ſegment thus found, and the two axes of the ellipſe continually together, and the product will give the area required.

EXAMPLES.

1. Required the area of the elliptic ſegment E A F, whoſe height A G is 10, and the axes of the ellipſe 2 A B and C D, 35 and 25 reſpectively.

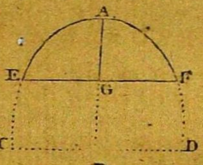
10.0000

---

* *Demon.* Let the tranſverſe diameter 2 A B $=t$, the conjugate C D $=c$, A G $=x$, and E G $=y$; then by the property of the curve we ſhall have $y=\frac{c}{t}\sqrt{tx-x^2}$, and the fluxion of the area E A F $=(y\dot{x})=\frac{c}{t}\times\dot{x}\sqrt{tx-x^2}$. But $\dot{x}\times\sqrt{tx-x^2}$ is known to expreſs the fluxion of the correſponding circular ſegment, whoſe verſed ſine is x, and the diameter t. Let the fluent of this expreſſion therefore be denoted by A, and then the fluent of $\frac{c}{t}\times\dot{x}\sqrt{tx-x^2}$ will be $=\frac{c}{t}\times$ A, from whence the rule is derived. Q. E. I.

Coroll. The ellipſe is equal to a circle whoſe diameter is a mean proportional between the two axes, and from hence the rule is formed for the whole ellipſe.

The



Here $\frac{10.0000}{35} = \frac{2.0000}{7} = .2857 =$ *tabular versed sine.*

And the tabular segment belonging to this is .185153.

Whence .185153 × 35(2AB) × 25 (CD) = 6.480355 × 25 = 162.0210 = *area of the segment required.*

2. What is the area of an elliptic segment cut off by a double ordinate parallel to the conjugate axe, at the distance of 36 from the centre, the axes being 120 and 40? *Ans.* 536.7504.

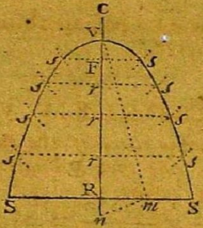
3. What is the area of a segment, cut off by an ordinate parallel to the transverse diameter, whose height is 5, the axes being 35 and 25? *Ans.* 97.845725.

PROBLEM VI.

To describe a parabola, any ordinate to the axe and its abscissa being given.

The area of an elliptic segment may also be found by the following rule.

Find the corresponding segment of the circle described upon the same axe to which the base of the segment is perpendicular. Then as this axe is to the other axe, so is the circular segment to the elliptic segment.



* *Construction.* 1. Bisect the given ordinate R S in m; join V m, and draw mn perpendicular to it, meeting the axe in n.

2. Make V C and V F each equal to R n, and F will be the focus of the curve.

3. Take any number of points r, r, &c. in the axe, through which draw the double ordinates S r S, S r S, &c. of an indefinite length.

4. With the radii C F, C r, &c. and centre F, describe arcs cutting the corresponding ordinates in the points s, s, &c. and the curve S V S drawn through

---

* Since Vmn is a right angled triangle, and mR is perpendicular to Vn, $VR \times Rn = VR \times VF = Rm^2 = \frac{1}{4}RS^2$, which is a known property of the parabola when F is the focus. And, because $sF^2 = Cr^2 - rF^2 = \overline{Cr + rF} \times \overline{Cr - rF} = \overline{Cr + rF} \times CF = 2Vr \times 2VF = Vr \times 4VF$, therefore s is a point in the curve of a parabola, and the same may be shewn of any other point s.

Besides the methods above, for finding the focus, it may be found arithmetically as follows:

Divide the square of the ordinate by 4 times the abscissa, and the quotient will be the focal distance vF.

• Several other methods of doing this, as well as of describing the curve itself, may also be found in Emerson's Conic Sections, and other performances.

all

all the points of interſection will be the parabola required.

Note. The line *s* F *s* paſſing through the focus F is called the parameter.

PROBLEM VII.

In a parabola, any three of the four following terms being given, viz. any two ordinates and their two abſciſſas, to find the fourth.

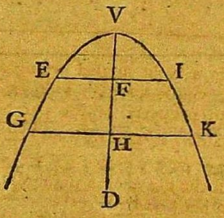
RULE.*

As any abſciſſa is to the ſquare of its ordinate, ſo is any other abſciſſa to the ſquare of its ordinate.

Or as the ſquare root of any abſciſſa is to its ordinate, ſo is the ſquare root of any other abſciſſa to its ordinate.

EXAMPLES.

1. The abſciſſa V F is 9, and its ordinate E F 6; required the ordinate G H, whoſe abſciſs V H is 16.



* If *x* and X be any two abſciſſas, and *y* and Y their correſponding ordinates, the equation of the curve will be $x\,Y^2 = \text{Cap.}\,X.\,y^2$, which is the ſame as the rule.

 Here

Here $\sqrt{9}$ ($\sqrt{\text{VF}}$) : 6 (EF) :: $\sqrt{16}$ ($\sqrt{\text{VH}}$) : $\frac{6 \times \sqrt{16}}{\sqrt{9}} = \frac{6 \times 4}{3} = \frac{24}{3} = 8 =$ *ordinate* GH.

Or,

9 (VF) : 36 (EF^2) :: 16 (VH) : $\frac{16 \times 36}{9} = 16 \times 4 = 64 = \text{GH}^2$, or $8 = \text{GH}$ *as before.*

2. The two abſciſſas are 9 and 16, and their correſponding ordinates 6 and 8; from any three of theſe to find the fourth.

PROBLEM VIII.

To find the length of any arc of a parabola, cut off by a double ordinate.

RULE.*

To the ſquare of the ordinate add $\frac{4}{3}$ of the ſquare of the abſciſſa, and twice the ſquare root of the ſum will be the length of the curve required.

EXAM-

---

* *Demon.* Let $x =$ any abſciſſa, $y =$ its ordinate, $a = \frac{1}{2}$ the parameter of the axe, and $q = \frac{y}{a}$. Then it is ſhewn by the writers on fluxions, that $aq\sqrt{1+q^2} + a \times$ hyp. log. of $(q + \sqrt{1+q}) = 2y \times : 1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} - \frac{3.5q^8}{2.4.6.8.9}$ &c. $= c =$ length of the curve. But $\sqrt{1 + \frac{1}{3}q^2} = 1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.9} + \frac{3q^6}{2.4.6.27}$ &c. agreeing with the former in the two firſt terms.

Therefore

EXAMPLES.

1. The abscissa VH is 2, and its ordinate GH 6: what is the length of the arc GVK?

Here 2^2 (VH²) $\times \frac{4}{3} + 36$ (GH²) $= \frac{4 \times 4}{3} + 36 = \frac{16}{3} + 36 = 5.333$, &c. $+ 36 = 41.333333$.

```
And 41.3333333(6.429
    36               2
    ----        ------
124)533         12.858 = length of the arc.
    496
    ----
1282)3733
     2564
     ---------
12849)116933
      115941
      ------
        1292
```

PROBLEM IX.

To find the area of a parabola, its base and height being given.

RULE.*

Multiply the base by the height, and $\frac{2}{3}$ of the product will be the area required.

EXAM-

Therefore $\frac{1}{2}\frac{c}{y} = \sqrt{1 + \frac{1}{3}q^2}$ *nearly*; and consequently $c = 2y\sqrt{1 + \frac{1}{3}q^2} = \sqrt{y^2 + \frac{4}{3}x^2}$ the same as the rule. Q. E. D.

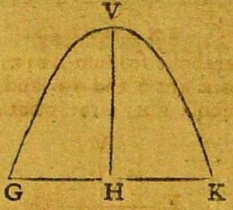
* *Demon.* Let VH $= x$, GH $= y$, and the parameter $= p$.

Then $px = y^2$ or $\sqrt{px} = y$ by the nature of the curve.

Whence

EXAMPLES.

1. What is the area of the parabola GVK, whose height VH is 12, and the base or double ordinate GK 16?



Here 16 (GK) × 12 (VH) × $\frac{2}{3}$ = $\frac{16 \times 12 \times 2}{3}$ = 16 × 4 × 2 = 128 = *area required.*

2. The abscissa is 12, and the double ordinate or base 38: what is the area? *Ans.* 304.

3. What is the area of a parabola whose abscissa is 10, and ordinate 8? *Ans.* $106\frac{2}{3}$.

PROBLEM X.

To find the area of the frustum of a parabola.

Whence the fluxion of the area ($= y\dot{x}$) $= \dot{x}\sqrt{px}$, and its fluent $= \frac{2}{3}x \times \sqrt{px}$.

But because $y = \sqrt{px}$, therefore $\frac{2}{3}x \times \sqrt{px} = \frac{2}{3}xy =$ area of the parabola, which is the same as the rule.

Coroll. Every parabola is $= \frac{2}{3}$ of its circumscribing parallelogram.

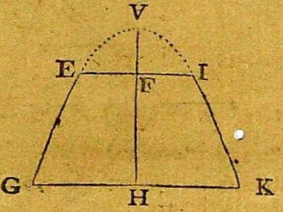
RULE.

RULE.*

Divide the difference of the cubes of the two ends of the frustum by the difference of their squares and this quotient multiplied by $\frac{2}{3}$ of the altitude, will give the area required.

EXAMPLES.

1. In the parabolic frustum GEIK, the two parallel ends EI and GK are 6 and 10, and the altitude, or part of the abscissa FH, is 3: what is the area?



Here $\overline{10^3-6^3}$ $(GK^3-EI^3) \div \overline{10^2-6^2}$ (GK^2-EI^2) $=\frac{10^3-6^3}{10^2-6^2}=\frac{1000-216}{100-36}=\frac{784}{64}=\frac{98}{8}=\frac{49}{4}=12.25.$

And $12.25\times\frac{2\times3}{3}=12.25\times2=24.5=$ *area required.*

---

* *Demon.* Let $D=GK$, $d=EI$, and $a=FH$.

Then by the nature of the curve $D^2-d^2 : a :: D^2 : \frac{aD^2}{D^2-d^2}=VH$, and $D^2-d^2 : a :: d^2 : \frac{ad^2}{D^2-d^2}=VF$.

And therefore $\frac{2}{3}\times\frac{aD^2}{D^2-d}-\frac{2}{3}\times\frac{ad^2}{D^2-d^2}=\frac{2}{3}a\times\frac{D^3-d^3}{D^2-d^2}=$ area of the frustum. Q. E. D.

 2. The

2. The greater end of the frustum is 24, the lesser end is 20, and their distance $5\frac{1}{2}$: what is the area?

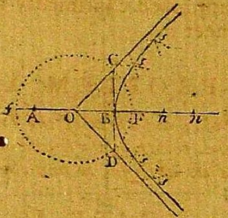
Ans. 121.3.

3. Required the area of the parabolic frustum, the greater end of which is 10, the less 6, and the height 4.

Ans. $32\frac{2}{3}$.

PROBLEM XI.

To construct an hyperbola, the transverse and conjugate diameters being given.



* *Construction.* 1. Make A B the transverse diameter, and C D perpendicular to it, the conjugate.

Note. Any parabolic frustum, is equal to a parabola of the same altitude, whose base is equal to one end of the frustum, increased by a third proportional to the sum of the two ends and the other end.

* The sum of two lines drawn from the foci of an ellipse to any point in the curve, is equal to its transverse diameter.

In like manner the difference of two lines drawn from the foci of any hyperbola to any point in the curve, is equal to its transverse diameter; as is shewn by the writers on *conics*.

But the arcs intersecting each other in *s*, *s*, &c. are described from the foci *f* and F, and with the distances A *n* and B *n*, whose difference is A B, and therefore the points *s*, *s*, &c. are in the curve of an hyperbola.

2. Bisect

2. Bisect A B in O, and from O with the radius O C, or O D, describe the circle D f C F, cutting A B produced in F and f, which points will be the two foci.

3. In A B produced take any number of points, n, n, &c. and from F and f, as centers, with the distances B n, A n, as radii, describe arcs cutting each other in s, s, &c.

4. Through the several points s, s, &c. draw the curve s B s, and it will be the hyperbola required.

5. If straight lines be drawn from the point O, through the extremities C D of the conjugate axis, they will be the asymptotes of the hyperbola, whose property it is to approach continually to the curve, and yet never to meet it.

PROBLEM XII.

In an hyperbola, any three of the four following terms being given, viz. the transverse and conjugate diameters, an ordinate and its abscissa, to find the fourth.

CASE I.

The transverse and conjugate diameters, and the two abscissas being known, to find the ordinate.

RULE.*

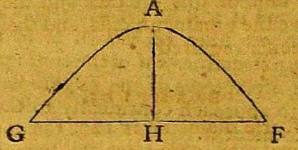
As the transverse diameter,
Is to the conjugate;
So is the square root of the product of the two abscissas,
To the ordinate required.

---

* Let t = transverse diameter, c = conjugate, x = abscissa, and y = ordinate. Then the general property of the curve is $t^2 : c^2 :: x \times (t+x) : y^2$; and from this analogy all the cases of this problem are deduced.

EXAMPLES.

1. In the hyperbola GAF, the tranſverſe diameter is 120, the conjugate 72, and the abſciſſa AH is 40: required the ordinate FH.



120 (*tranſ.*) : 72 (*conj.*) :: $\sqrt{}(160 \times 40)$: $\frac{72 \times \sqrt{}(160 \times 40)}{120} = \frac{6 \times \sqrt{}(160 \times 40)}{10} = \frac{3}{5}\sqrt{}(160 \times 40) = \frac{3}{5}\sqrt{}6400 = \frac{3}{5} \times 80 = \frac{3 \times 80}{5} = 3 \times 16 = 48 =$ *ordinate* FH.

2. The tranſverſe diameter is 24, the conjugate 21, and the leſſer abſciſſa 8: what is its ordinate?

Anſ. 14.

3. The tranſverſe diameter of an hyperbola is 50, the conjugate 30, and the greater abſciſſa 12: required the ordinate.

CASE II.

The tranſverſe and conjugate diameters, and an ordinate being given, to find the two abſciſſas.

Note. In the hyperbola, the leſs abſciſſa added to the axis, gives the greater.

RULE.

RULE.

As the conjugate diameter is to the tranſverſe,
So is the ſquare root of the ſum of the ſquares of the ordinate and ſemi-conjugate,
To the diſtance between the ordinate and the centre, or half the ſum of the abſciſſas.

Then the ſum of this diſtance and the ſemi-tranſverſe will give the greater abſciſſa, and their difference the leſſer abſciſſa required *.

EXAMPLES.

The tranſverſe diameter is 120, the conjugate 72, and the ordinate 48: what are the two abſciſſas?

1296 = *ſquare of the ſemi-conjugate.*
2304 = *ſquare of the ordinate.*

3600(60 = *ſquare root.*
36

00

As 72 : 120 : : 60
120

72)7200(100 = $\frac{1}{2}$ *ſum of the abſciſſas.*
72 60 = *ſemi-tranſverſe.*

160 = *greater abſciſſa.*

40 = *leſſer abſciſſa.*

* This rule in ſpecies is $\frac{t}{c}\sqrt{\frac{1}{4}c^2+y^2} \pm \frac{1}{2}t = x$, = greater or leſs abſciſſa, according as the upper or under ſign is uſed.

2. The

2. The transverse and conjugate diameters are 24 and 21: required the two abscissas to the ordinate 14.

Ans. 32 *and* 8.

3. The transverse being 60, and the conjugate 36; required the two abscissas to the ordinate 24.

Ans. 80 *and* 20.

CASE III.

The transverse diameter, the two abscissas and the ordinate being given, to find the conjugate.

RULE.

As the square root of the product of the two abscissas,
Is to the ordinate;
So is the transverse diameter,
To the conjugate *.

EXAMPLES.

1. The transverse diameter is 120, the ordinate is 48, and the two abscissas are 160 and 40: required the conjugate.

```
160
 40
---
6400(80
64
---
 00
```

* This rule expressed algebraically is $ty \div \sqrt{x \times (t+x)} \times x = c =$ conjugate diameter.

As

As 80 : 48 :: 120 *the transverse axis.*

$$\begin{array}{r} 48 \\ \hline 960 \\ 480 \\ \hline 8.0)576.0 \\ \hline \end{array}$$

72 = *conjugate required.*

2. The transverse diameter is 24, the ordinate 14, and the abscissas 8 and 32: required the conjugate.

Ans. 21.

CASE IV.

The conjugate diameter, the ordinate and two abscissas being given, to find the transverse.

RULE.

1. Add the square of the ordinate to the square of the semi-conjugate, and find the square root of their sum.

2. Take the sum or difference of the semi-conjugate and this root, according as the less or greater abscissa is used, and then say,

As the square of the ordinate,
Is to the product of the abscissa and conjugate;
So is the sum, or difference, above found,
To the transverse required *.

---

* This rule in algebraic terms is $\frac{cx}{y^2} \times (\sqrt{\frac{1}{4}c^2 + y^2} \pm \frac{1}{2}c) = t$ = transverse diameter.

EXAM-

EXAMPLES.

1. The conjugate diameter is 72, the ordinate is 48, and the lesser abscissa 40: what is the transverse?

Here $\sqrt{48^2+36^2}=\sqrt{2304+1296}=\sqrt{3600}=60$, *and* $60+36=96$.

Also $72\times40=2880=$ *product of the abscissa and conjugate. Whence.*

As 2304 : 2880 :: 96

```
          2880
            96
         -----
         17280
        25920
        ------
2304)276480(120=transverse required.
      2304
      ----
       4608
       4608
       ----
```

2. The conjugate diameter is 21, the lesser abscissa 8, and its ordinate 14: required the transverse.

Ans. 24.

3. Required the transverse diameter of the hyperbola, whose conjugate is 36, the lesser abscissa being 20.

Ans. 60.

PROBLEM XIII.

To find the length of any arc of an hyperbola, beginning at the vertex.

RULE.*

1. To 19 times the tranſverſe, add 21 times the parameter of the axis, and multiply the ſum by the quotient of the abſciſſa divided by the tranſverſe.

2. To 9 times the tranſverſe, add 21 times the parameter, and multiply the ſum by the quotient of the abſciſſa divided by the tranſverſe.

3. To each of the products, thus found, add 15 times the parameter, and divide the former by the latter; then this quotient being multiplied by the ordinate will give the length of the arc *nearly*.

EXAMPLES.

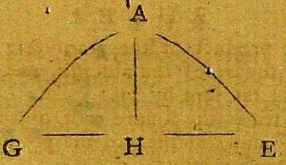
1. In the hyperbola GAE, the tranſverſe diameter is 80, the conjugate 60, the ordinate GH 10, and the abſciſſa AH 2.1637: required the length of the arc AG.

---

* *Demon.* Let $t =$ ſemi-tranſverſe axe, $c =$ ſemi-conjugate, $x =$ ordinate, and $y =$ abſciſſa. Then will $y \times (1 + \frac{t^2}{6c^4} y^2 - \frac{t^4 + 4t^2c^2}{40c^8} y^4 + \frac{t^6 + 4t^4c^2 + 8t^2c^4}{112c^{12}} y^6$, &c.$) =$ length of the arc, as is ſhewn by the writers on fluxions.

But $x = \frac{t}{c}\sqrt{c^2 + y^2} - a$, and $\frac{2c}{t} =$ parameter $= p$, by the nature of the curve. Conſequently the rule becomes $(15p + \overline{19t + 21p} \times \frac{x}{t}) \div (15p + \overline{9t + 21p} \times \frac{x}{t}) \times y = (15pt + 19tx + 21px) \div (15pt + 9tx + 21px) \times y = y \times : 1 + \frac{2x}{3p} - \frac{2x^2}{5p}$ &c. which by ſubſtituting the values of x and p, and expanding the terms, gives a ſeries, agreeing nearly in the three firſt terms with the former, and therefore the rule is an approximation.

If



Here $80 : 60 :: 60 : \frac{60\times60}{80} = \frac{3\times60}{4} = 3\times15 = 45$ = *parameter.*

And $\overline{80\times19\times45\times21}\times\frac{2.1637}{80} = \overline{1520+945}\times .02704 = 2465\times.02704 = 66.6536.$

Also $\overline{80\times9+45\times21}\times\frac{2.1637}{83} = \overline{720+645}\times.0274 = 1665\times.02704 = 45.0216.$

Whence $\overline{675+66.6536}\div\overline{675\times45.0216} = 741.6536 \div 720.0216 = 1.03004$; *and* $1.03004\times10 = 10.3004$ = *length of the arc required.*

2. The tranſverſe diameter of an hyperbola is 120, the conjugate 72, the ordinate 48, and the abſciſſa 40: required the length of the curve.

Anſ. 62.6496.

3. Required the whole length of the curve of an hyperbola, to the ordinate 10; the tranſverſe and conjugate axes being 10 and 60. *Anſ.* 20.601.

PROBLEM XIV.

To find the area of an hyperbola, the tranſverſe, conjugate and abſciſſa being given.

If t = ſemi-tranſverſe, c = ſemi-conjugate, and y = ordinate drawn from the end of the required arc; then $y\times(1+\frac{t^2y^2}{6c^4}\text{A} - \frac{t^2+4c^2}{c^2}.\frac{3y^2}{20}\text{B} + \frac{t^4+4t^2c^2+8c^4}{t^2+4c^2}.\frac{5y^2}{14c^4}\text{C}$ &c.) = length of the arc.

RULE.

RULE.*

1. To the product of the transverse and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.

2. Add 4 times the square root of the product of the transverse and abscissa, to the product last found, and divide the sum by 75.

3. Divide 4 times the product of the conjugate and abscissa by the transverse, and this last quotient multiplied by the former will give the area required *nearly*.

EXAMPLES.

1. In the hyperbola GAE, the transverse axis is 30, the conjugate 18, and the abscissa or height AH is 10: what is the area?

---

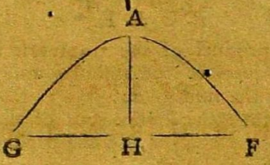
* *Demon.* Let $t =$ transverse diameter, $c =$ conjugate, $x =$ abscissa, $y =$ ordinate, and $z = \frac{x}{t2 + x}$. Then it is well known that $4xy \times (\frac{1}{3} - \frac{1}{1 \cdot 3 \cdot 5} z - \frac{1}{3 \cdot 5 \cdot 7} z^2 - \frac{1}{5 \cdot 7 \cdot 9} z^3$, &c. $=$ area of the hyperbola.)

But $\frac{ty}{\sqrt{ta + x^2}} = c =$ conjugate axe, by the nature of the hyperbola. Consequently the expression for the rule $=$
$$\frac{4cx}{t} \times \frac{21\sqrt{tx + \frac{5}{7}x} + 4\sqrt{tx}}{75} = 4xy \times \frac{21\sqrt{tx + \frac{5}{7}x^2} + 4\sqrt{tx}}{\sqrt{tx - x^2}}$$
And this thrown into a series will very nearly agree with the former; which shews the rule to be an approximation.

Q. E. I.

Rule 2. If $2Y$, $2y =$ bases, V, and v their distances from the centre, and the other letters as before, then will $VY - vy - \frac{tc}{4} \times$ hyp. log. of $\frac{tY + cV}{ty + cv} =$ area of the frustum of the hyperbola.

30



Here $21\sqrt{30\times10+\frac{5}{7}\times10^2}=21\sqrt{300+500\div7}=$ $21\sqrt{300+71.42857}=21\sqrt{371.42857}=21\times19.272$ $=404.712.$

And $(4\sqrt{30\times10}+404.712)\div75=(4\sqrt{400}+$ $404.712)\div75=(4\times17.3205+404.712)\div75=$ $(69.282+404.712)\div75=473.994\div75=6.3199.$

Whence $\frac{18\times19\times4}{30}\times6.3199=\frac{18\times4}{3}\times6.3199=6$ $\times4\times6.3199=24\times6.3196=151.6776=$ *area required.*

2. The tranſverſe diameter is 100, the conjugate 60, and the leſſer abſciſs 50; what is the area of the hyperbola? *Anſ.* 3220.363472.

3. Required the area of the hyperbola to the abſciſſa 25, the two axes being 50 and 30. *Anſ.* 805.0909.

Rule 3. If t be put = tranſverſe axis, c = conjugate, and x = abſciſſa, the area of a ſegment of an hyperbola, cut off by a double ordinate, will be $=\frac{4\sqrt{tx+\frac{3}{4}x^2}+\sqrt{tx}}{15}\times\frac{4cx}{t}$ extremely near.

OF

OF THE

MENSURATION

OF

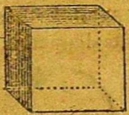
SOLIDS.

DEFINITIONS.

1. THE *meaſure* of any ſolid body, is the whole capacity or content of that body, when conſidered under the triple dimenſions of length, breadth, and thickneſs.

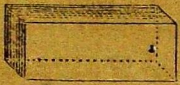
2. A *cube* whoſe ſide is one inch, one foot, or one yard, &c. is called the *meaſuring unit*; and the content or ſolidity of any figure, is computed by the number of thoſe cubes contained in that figure.

3. A *cube* is a ſolid contained by ſix equal ſquare ſides.



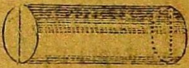
4. A *parallelopipedon* is a ſolid contained by ſix quadrilateral planes, every oppoſite two of which are equal and parallel.

5. A

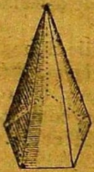


5. A *priſm* is a ſolid whoſe ends are two equal, parallel, and ſimilar plane figures, and its ſides parallelograms.

6. A *cylinder* is a ſolid deſcribed by the revolution of a right angled parallelogram about one of its ſides, which remains fixed.



7. A * *pyramid* is a ſolid whoſe ſides are all triangles meeting in a point at the vertex, and the baſe any plane figure whatever.



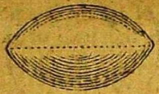
8. A *ſphere* is a ſolid deſcribed by the revolution of a ſemi-circle about its diameter, which remains fixed.

* The definition of a cone has been given already.

9. The centre of a ſphere is a point within the figure, every where equally diſtant from the convex ſurface of it.

10. The diameter of the ſphere is a ſtraight line paſſing through the centre, and terminated both ways by the convex ſuperficies.

11. A *circular ſpindle* is a ſolid generated by the revolution of a ſegment of a circle about its chord, which remains fixed.

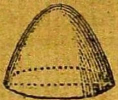


12. A *ſpheroid* is a ſolid generated by the revolution of a ſemi-ellipſis about one of its diameters, which is conſidered as quieſcent.

The ſpheroid is called *prolate*, when the revolution is made about the tranſverſe diameter, and *oblate* when it is made about the conjugate diameter.

13. *Elliptic*, *parabolic*, and *hyperbolic ſpindles*, are generated in the ſame manner as the circular ſpindle, the double ordinate of the ſection being always fixed or quieſcent.

14. *Parabolic* and *hyperbolic conoids*, are ſolids formed by the revolution of a ſemi-parabola or hyperbola about its tranſverſe axis, which is conſidered as quieſcent.



15. The

15. The *segment* of a pyramid, ſphere, or any other ſolid, is a part cut off from the top by a plane parallel to the baſe of that figure.

16. A *fruſtum* or *trunk*, is the part that remains at the bottom after the ſegment is cut off.

17. The *zone of a ſphere*, is that part which is intercepted between two parallel planes; and when thoſe planes are equally diſtant from the centre, it is called the middle zone of the ſphere.

18. The height of a ſolid is a perpendicular, drawn from its vertex to the baſe or plane on which it is ſuppoſed to ſtand.

PROBLEM I.

To find the ſolidity of a cube, the height of one of its ſides being given.

RULE.*

Multiply the ſide of the cube by itſelf, and that product again by the ſide, and it will give the ſolidity required.

---

* *Demon.* Conceive the baſe of the cube to be divided into a number of little ſquares, each equal to the *ſuperficial meaſuring unit.*

Then will thoſe ſquares be the baſes of a like number of ſmall cubes, which are each equal to the *ſolid meaſuring unit.*

But the number of little ſquares contained in the baſe of the cube are equal to the ſquare of the ſide of that baſe, as has been ſhewn already.

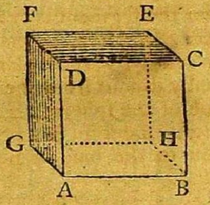
And conſequently, the number of ſmall cubes contained in the whole figure, muſt be equal to the ſquare of the ſide of the baſe multiplied by the height of that figure; or, which is the ſame thing, the ſquare of the ſide of the baſe multiplied by the baſe, is equal to the ſolidity. Q. E. D.

Note. The ſurface of the cube is equal to 6 times the ſquare of its ſide.

EXAM-

EXAMPLES.

1. The side A B, or B C of the cube A B C D F G H E, is 25.5: what is the solidity?



Here $AB^3 = \overline{25.5}|^3 = 25.5 \times 25.5 \times 25.5 = 25.5 \times 650.25 = 16581.375$ *the answer, or content of the cube.*

2. The side of a cube is 15 inches: what is the solidity?

	fo.	in.	pa.
Ans.	1	11	5

3. What is the solidity of a cube whose side is 17.5 inches? *Ans.* 3.101 *feet.*

PROBLEM II.

To find the solidity of a parallelopipedon.

RULE.*

Multiply the length by the breadth, and that product again by the depth or altitude, and it will give the solidity required.

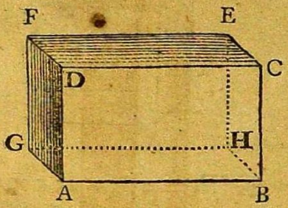
---

* The reason of this rule, as well as of the following ones for the prism and cylinder, is the same as that for the cube.

Note. The surface of the parallelopipedon is equal to the sum of the areas of each of its sides or ends.

EXAMPLES.

1. Required the ſolidity of the parallelopipedon ABCDFEHG, whoſe length AB is 8 feet, its breadth FD $4\frac{1}{2}$ feet, and the depth or altitude AD $6\frac{3}{4}$ feet?



Here AB × AD × FD = 8 × 6.75 × 4.5 = 54 × 4.5 = 243 *ſolid feet, the contents of the parallelopipedon required.*

2. The length of a parallelopipedon is 15 feet, and each ſide of its ſquare baſe 21 inches: what is the ſolidity? *Anſ.* 45.9 *feet.*

3. What is the ſolidity of a block of marble, whoſe length is 10 feet, its breadth $5\frac{3}{4}$ feet, and the depth $3\frac{1}{2}$ feet? *Anſ.* 201.25 *feet.*

PROBLEM III.

To find the ſolidity of a priſm.

RULE.*

Multiply the area of the baſe into the perpendicular height of the priſm, and the product will be the ſolidity.

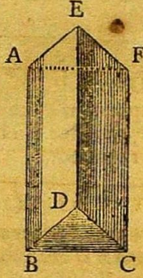
---

* The ſurface of a priſm, is equal to the ſum of the areas of the two ends and each of its ſides.

EXAM-

EXAMPLES.

1. What is the solidity of the triangular prism ABCFED, whose length AB is 10 feet, and either of the equal sides BC, CD, or DB, of one of its equilateral ends BCD, $2\frac{1}{2}$ feet?



Here $\frac{1}{4} \times 2.5^2 \times \sqrt{3} = \frac{1}{4} \times 6.25 \times \sqrt{3} = 1.5625 \times \sqrt{3} = 1.5625 \times 1.732 = 2.70625$ = area of the base BCD.

Or, $\frac{2.5 + 2.5 + 2.5}{2} = \frac{7.5}{2} = 3.75 = \frac{1}{2}$ sum of the sides, BC, CD, DB *of the triangle* CDB.

And $3.75 - 2.5 = 1.25$, ∴ 1.25, 1.25 and 1.25 = 3 differences.

Whence $\sqrt{3.75 \times 1.25 \times 1.25 \times 1.25} = \sqrt{3.75 \times 1.25^3} = \sqrt{7.32521875} = 2.7063$ = area of the base as before.

And $2.7063 \times 10 = 27.063$ solid feet, the content of the prism required.

2. What is the solidity of a triangular prism, whose length is 18 feet, and one side of the equilateral end $1\frac{1}{2}$ feet? *Ans.* 17.50859 *feet.*

---

* This rule may be seen in the Notes to Problem III. p. 48.

 Re-

This Rev

3. Required the ſolidity of a priſm whoſe baſe is a hexagon, ſuppoſing each of the equal ſides to be 1 foot 4 inches, and the length of the priſm 15 feet.

Anſ. 69.282 *feet.*

PROBLEM IV.

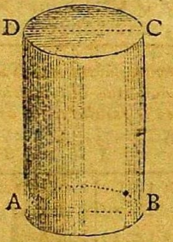
To find the convex ſurface of a cylinder.

RULE.*

Multiply the periphery or circumference of the baſe, by the height of the cylinder, and the product will be the convex ſurface required.

EXAMPLES.

1. What is the convex ſurface of the right cylinder A B C D, whoſe length B C is 20 feet, and the diameter of its baſe A B 2 feet?



* *Demon.* If the periphery of the baſe be conceived to move, in a direction parallel to itſelf, it will generate the convex ſuperficies of the cylinder; and therefore the ſaid periphery being multiplied by the length of the cylinder, will be equal to that ſuperficies. Q. E. D.

Note. If twice the area of either of the ends be added to the convex ſurface, it will give the whole ſurface of the cylinder.

Here

Here 3.14159×2=6.28318=*periphery of the base* A. B.

And 6.28318×20=125.6636 *square feet, the convexity required.*

2. What is the convex surface of a right cylinder, the diameter of whose base is 30 inches, and the length 60 inches? *Ans.* 5654.862 *inches.*

3. Required the convex superficies of a right cylinder, whose circumference is 8 feet 4 inches, and its length 14 feet. *Ans.* 116.666 *&c. feet.*

PROBLEM V.

To find the solidity of a cylinder.

RULE.*

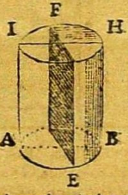
Multiply the area of the base by the perpendicular height of the cylinder, and the product will be the solidity.

EXAM-

---

* The four following cases contain all the rules for finding the superficies, and solidities of *cylindric ungulas.*

I. *When the section is parallel to the axis of the cylinder.*



Rule. 1. Multiply the length of the arc line of the base by the height of the cylinder, and the product will be the *curve surface.*

2. Multiply the area of the base by the height of the cylinder, and the product will be the *solidity.*

II. *When*

EXAMPLES.

1. What is the ſolidity of the cylinder A B C D, the diameter of whoſe baſe A B is 30 inches, and the height B C 50 inches?

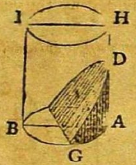
Here

II. *When the ſection paſſes obliquely through the oppoſite ſides of the cylinder.*

Rule. 1. Multiply the circumference of the baſe of the cylinder by half the ſum of the greateſt and leaſt lengths of the ungula, and the product will be the *curve ſurface.*

2. Multiply the area of the baſe of the cylinder by half the ſum of the greateſt and leaſt lengths of the ungula, and the product will be the *ſolidity.*

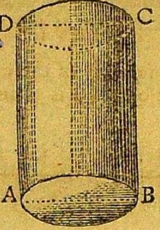
III. *When the ſection paſſes through the baſe of the cylinder, and one of its ſides.*



Rule. 1. Multiply the ſine of half the arc of the baſe by the diameter of the cylinder, and from this product ſubtract the product of the arc and coſine.

2. Multiply the difference, thus found, by the quotient of the height divided by the verſed ſine, and the product will be the *curve ſurface.*

3. From



Here $.7854 \times 30^2 = .7854 \times 900 = 706.86 =$ *area of the base* AB.

And $706.86 \times 50 = 35343$ *cubic inches; or* $\frac{35343}{1728}$ $= 20.4531$ *solid feet, the answer required.*

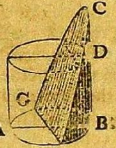
2. What

---

1. From $\frac{2}{3}$ of the cube of the right sine of half the arc of the base, subtract the product of the area of the base and the cosine of the said half arc.

2. Multiply the difference, thus found, by the quotient arising from the height divided by the versed sine, and the product will be the *solidity*.

IV. *When the section passes obliquely through both ends of the cylinder.*



Rule. 1. Conceive the section to be continued, till it meets the side of the cylinder produced; then say, as the difference of

2. What is the ſolidity of a cylinder whoſe height is 5 feet, and the diameter of the end 2 feet?

Anſ. 15.708 *feet.*

3. What is the ſolidity of a cylinder whoſe height is 20 feet, and the circumference of its baſe 20 feet alſo? *Anſ.* 636.64 *feet.*

4. The circumference of the baſe of an oblique cylinder is 20 feet, and the perpendicular height 19.318: what is the ſolidity? *Anſ.* 614.926 *feet.*

PROBLEM VI.

To find the convex ſurface of a right cone.

RULE.*

Multiply the circumference of the baſe by the ſlant height, or the length of the ſide of the cone, and half the product will be the ſurface required.

EXAM-

---

of the verſed ſines of half the arcs of the two ends of the ungula, is to the verſed ſine of half the arc of the leſſer end, ſo is the height of the cylinder to the part of the ſide produced.

2. Find the ſurface of each of the ungulas, thus formed, by caſe the third, and their difference will be the *ſurface required.*

3. In like manner find the ſolidities of each of the ungulas, and their difference will be the *ſolidity required.*

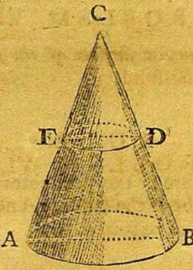
* *Demon.* Let AB $= a$, BC $= b$, $3.1416 = p$, and ED $= y$.

Then $a : b :: y : \frac{by}{a} =$ DC; and $py =$ circumference of the circle ED.

But

EXAMPLES.

1. The diameter of the baſe A B is 3 feet, and the ſlant height A C or B C 15 feet: required the convex ſurface of the cone A C B.



Here $3.1416 \times 3 = 9.4248 =$ *circumference of the baſe* A B.

And, $\frac{9.4248 \times 15}{2} = \frac{141.3720}{2} = 70.686$ *ſquare feet, the convex ſurface required.*

2. The diameter of a right cone is 4.5 feet, and the ſlant height 20 feet: required the convex ſurface.

Anſ. 141.372 *feet.*

But $py \times \frac{b\dot{y}}{a} =$ fluxion of the ſurface of C E D, and its fluent $= \frac{pby^2}{2a}$; which, when $y = a$, becomes $\frac{pba}{2}$ = convex ſurface of the whole cone. Q. E. D.

To find the ſurface of a right pyramid.

Rule. Multiply the perimeter of the baſe by the length of the ſide, or ſlant height of the cone, and half the product will be the ſurface required.

 3. The

3. The circumference of the base is 10.75, and the slant height 18.25: what is the convex surface?

Ans. 98.09375.

PROBLEM VII.

To find the convex surface of the frustum of a right cone.

RULE.*

Multiply the sum of the perimeters of the two ends, by the slant height of the frustum, and half the product will be the surface required.

---

* *Demon.* Let the perimeter of the circle AB = P, that of ED = p, BD = b, and the rest as in the last problem.

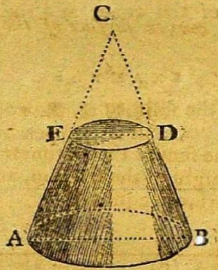
Then P : p :: b (BC) : CD; and, by division, P − p :: b − CD (b) : CD $= \frac{pb}{P-p}$; but P $\times (b + \frac{pb}{P-p})$ = twice the convex surface of the whole cone, by the last rule; and also $p \times \frac{pb}{P-p}$ = twice the convex surface of the part ECD. Therefore $P \times (b+\frac{pb}{P-p}) - p \times \frac{pb}{P-p} = bP + \overline{P-p} \times \frac{pb}{P-p} = bP + bp = \overline{P+p} \times b$ = twice the convex surface of the frustum ABDE; and the half thereof is $\frac{(P \times p) \times b}{2}$, which is the same as the rule. Q. E. D.

To find the surface of the frustum of a right pyramid.

Rule. Multiply the sum of the perimeters of the end by the slant height, and half the product will be the surface required.

EXAMPLES.

1. In the fruſtum A B E D, the circumferences of the two ends A B and D E are 22.5 and 15.75 reſpectively, and the ſlant height B D is 26: what is the convex ſurface?



Here $\frac{(22.5+15.75)\times 26}{2} = \overline{22.5+15.75}\times 13 =$ $38.25\times 13 = 497.25 =$ *convex ſurface required.*

2. What is the convex ſurface of the fruſtum of a right cone, the circumference of the greater end being 30 feet, that of the leſſer end 10 feet, and the length of the ſlant ſide 20 feet? *Anſ.* 400 *feet.*

3. What is the convex ſurface of the fruſtum of a right cone, the diameters of the ends being 8 and 4 feet, and the length of the ſlant ſide 20 feet?

Anſ. 376.992 *feet.*

4. If a ſegment of 6 feet ſlant height be cut off a cone whoſe ſlant height is 30 feet, and circumference of its baſe 10 feet: what is the ſurface of the fruſtum?

Anſ. 144 *feet.*

PROBLEM VIII.

To find the solidity of a cone or pyramid.

RULE*.

Multiply the area of the base by $\frac{1}{3}$ of the perpendicular height of the cone, and the product will be the solidity.

EXAMPLES.

1. Required the solidity of the cone A C B, whose diameter A B is 20, and its perpendicular height C S 24.

---

* *Demon.* Let S C $= a$, C $s = x$, and A $=$ area of the base of the cone A C B.

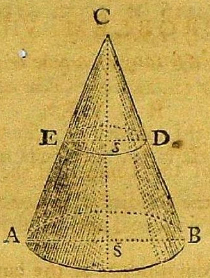
Then a^2 (C S^2) : x^2 (C s^2) : : AB^2 : ED^2 (by sim. $\triangle s$) : : A : $\frac{Ax^2}{a}$, ($=$ area of the circle E D) because all circles are as the squares of their diameters.

But $\frac{Ax^2}{a^2} \times \dot{x} =$ fluxion of the cone E C D, and its fluent $= \frac{Ax^3}{3a^2}$; which, when $x = a$, becomes $\frac{Aa}{3} = A \times \frac{a}{3}$ for the solidity of the whole cone. Q. E. D.

In the pyramid C E D B it will be a^2 (C S^2) : x^2 (C s^2) : : CE^2 : ce^2 : : ED^2 : eo^2 (by sim. $\triangle s$) : : A (area of the base E B) : $\frac{Ax^2}{a^2}$ (area of the polygon eb) because all similar figures are as the squares of their like sides.

But $\frac{Ax^2}{a^2} \times \dot{x} =$ fluxion of the pyramid $ceob$, and its correct fluent $= A \times \frac{a}{3}$ the same as in the cone; and this rule is general, let the figure of the base be what it will.

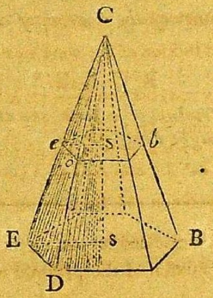
Here



Here $.7854 \times 20^2 = .7854 \times 400 = 314.16 =$ *area of the base* A B.

And $314.16 \times \frac{24}{3} = 314.16 \times 8 = 2513.28 =$ *solidity required.*

2. Required the ſolidity of the hexagonal pyramid E C B D, each of the equal ſides of its baſe being 40, and the perpendicular height C S 60.



Here 2.598076 (*multiplier when the ſide is* 1) $\times 40^2 =$ $2.598076 \times 1600 = 4156.9216 =$ *area of the baſe.*

And,

And, $4156.9216 \times \frac{60}{3} = 4156.9216 \times 20 =$ 83138.432 *solidity required.*

3. Required the solidity of a triangular pyramid, whose height is 30, and each side of the base 3.

Ans. 38.97117.

4. Required the solidity of a square pyramid, each side of whose base is 30, and the perpendicular height 20. *Ans.* 6000.

5. What is the solidity of a cone, the diameter of whose base is 18 inches, and its altitude 15 feet?

Ans. 8.83575 *feet.*

6. If the circumference of the base of a cone be 40 feet, and the height 50 feet; what is the solidity?

Ans. 2120 *feet.*

7. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?

Ans. 27.527.

PROBLEM IX.

To find the solidity of the frustum of a cone or pyramid.

RULE.*

1. *For the frustum of a cone, the diameters of the two ends, and the height being given.*

Divide

---

* *Demon.* Let D = diameter AB, d = ED, p = .7854, and h = ss = height of the frustum of the cone ABDE, (see the last figure.)

Then D : d :: BS : Cs, and D − d : d :: Cs − Cs (h) : $\frac{dh}{D-d}$ = Cs = height of the cone EDC.

But

Divide the difference of the cubes of the diameters of the two ends, by the difference of the diameters, and this quotient being multiplied by .7854 and again by $\frac{1}{3}$ of the height will give the solidity.

2. *For the fruſtum of a pyramid, the ſides and height being given.*

To the areas of the two ends of the fruſtum add the ſquare root of their product, and this ſum being multiplied by $\frac{1}{3}$ of the height will give the ſolidity.

1. What is the ſolidity of the fruſtum of the cone EABD, the diameter of whoſe greater end AB is

---

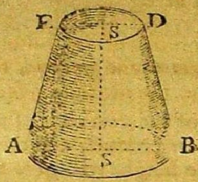
But $\frac{pD^2}{3} \times (b + \frac{db}{D-d}) =$ ſolidity of the whole cone ACB, and $\frac{pd^2}{3} \times \frac{db}{D-d} =$ that of the cone ECD. Therefore $\frac{pD^2}{3} \times (b + \frac{db}{D-b}) - \frac{pd^2}{3} \times \frac{db}{D-d} = (D^2 \times \frac{Db}{D-d} - d^2 \times \frac{db}{D-d}) \times \frac{p}{3} = (bD^2 + \overline{D^2 - d^2} \times \frac{db}{D-d}) \times \frac{p}{3} = \frac{D3 - d3}{D-d} \times \frac{bp}{3} =$ ſolidity of the fruſtum ABDE. Q. E. D.

Again, for the polygon, let $S = ED$, $s = ed$, and $m =$ proper multiplier in the table of polygons; then $S : s :: cS : cs$, and $S - s : s :: cS - cs\ (b) : \frac{bs}{S-s}$.

But mS^2 and ms^2 are the areas of polygons whoſe ſides are S and s reſpectively. And therefore $\frac{mS^2}{3} \times (b + \frac{bs}{S-s}) - \frac{ms^2}{3} \times \frac{bs}{S-s} = (mS^2 + \overline{mS^2 - ms^2} \times \frac{bs}{S-s} \times \frac{b}{3} = (mS + mSs + ms^2) \times \frac{b}{3} =$ ſolidity of the fruſtum eEDBb, which is the ſame as the rule.

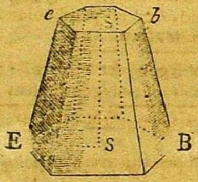
5 feet,

5 feet, that of the leſſer end E D 3 feet, and the perpendicular height *s* S 9 feet?



Here $\frac{5^3-3^3}{5-3}\times.7854\times\frac{9}{3}=\frac{125-27}{2}\times.7854\times3=$ $\frac{98}{2}\times2.3562=49\times2.3562=115.4538$ *ſolid feet, the content of the fruſtum.*

2. What is the ſolidity of the fruſtum *e* E D B *b*, of an hexagonal pyramid, the ſide E D of whoſe greater end is 4 feet, that *e d* of the leſſer end 3 feet, and the height S *s* 9 feet?



Here 2.598076 (*tab. mult.*) $\times 3^2 = 2.598076\times9=$ $23.382684=$ *area of the polygon e b.*

And 2.598076 (*tab. mult.*) $\times4^2=2.598076\times16=$ $41.569216=$ *area of the polygon* E B.

Whence

Whence $\sqrt{41.569216 \times 23.382684} = \sqrt{971.999841}$ $= 31.176911$.

And $(41.569216 + 23.382684 + 31.176911) \times \frac{9}{3} = 96.128811 \times 3 = 288.386433$ *feet*, $=$ *solidity required.*

3. What is the solidity of the frustum of a cone, the diameter of the greater end being 4 feet, that of the lesser end 2 feet, and the altitude 9 feet?

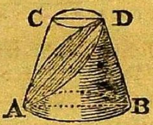
Ans. 65.9736.

4. What

---

* The following cases contain all the rules for finding the superficies and solidities of conical ungulas.

1. *When the section passes through the opposite extremities of the ends of the frustum.*

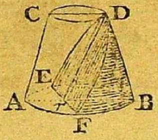


Let $D = AB$, the diameter of the greater end, $d = CD$ the diameter of the less end, $h =$ perpendicular height of the frustum, and $n = .78539$, &c.

Then will $\overline{D^2 - d\sqrt{Dd}} \div \overline{D - d} \times \frac{1}{3} n d h =$ solidity of the elliptic ungula ADB.

And $\frac{n}{D - d} \sqrt{\overline{4h^2 + (D - d^2)}} \times \overline{D^2 - (D - d)\sqrt{4Dd}} =$ curve surface of ADB.

2. *When the section cuts off part of the base, and makes the angle D r B less than the angle C A B.*



Let

4. What is the ſolidity of the fruſtum of a cone, the circumference of the greater end being 40, that of the leſſer end 20, and the length or height 50?

Anſ. 3713.64.

5. What is the ſolidity of the fruſtum of a ſquare pyramid, one ſide of the greater end being 18 inches, that of the leſſer end 15 inches, and the height 60 inches? *Anſ.* 16380 *inches*.

6. What is the ſolidity of the fruſtum of an hexagonal pyramid, the ſide of whoſe greater end is 3 feet, that of the leſſer end 2 feet, and the length 12 feet?

Anſ. 197.453776 *feet*.

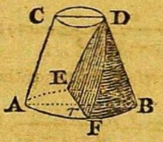
PRO-

Let S = tabular ſegment, whoſe verſed ſine is $\text{B}r \div \text{D}$, s = tab. ſeg. whoſe verſed ſine is $\overline{\text{B}r - (\text{D} - d)} \div d$, and the other letters as before.

Then $(\text{S} \times \text{D}^3 - s \times d^3 \times \frac{\text{B}r}{\text{B}r - \overline{\text{D} - d}} \sqrt{\frac{\text{B}r}{\text{B}r - \overline{\text{D} - d}}}) \times \frac{\frac{1}{3}b}{\text{D} - d}$ = ſolidity of the elliptic hoof EFBD.

And $\frac{1}{\text{D} - d} \sqrt{4b^2 + (\text{D} - d)^2} \times$ (ſeg. FBE $- \frac{d^2}{\text{D}^2} \times \frac{\frac{1}{2} \times (\text{D} + d) - \text{A}r}{d - \text{A}r} \times \sqrt{\frac{\text{B}r}{d - \text{A}r}} \times$ ſeg. of the circle AB, whoſe height is $\text{D} \times \frac{d - \text{A}r}{d}$ = convex ſurface of EFBD.

3. *When the ſection is parallel to one of the ſides of the fruſtum.*



Let A = area of the baſe FBE, and the other letters as before.

Then

PROBLEM X.

To find the solidity of a cuneus or wedge.

RULE.*

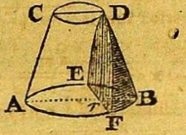
Multiply the sum of twice the length of the base and the length of the edge, by the product of the height

---

Then $(\frac{A \times D}{D - d} - \frac{4}{5} d \sqrt{(D-d) \times d}) \times \frac{1}{3} b =$ solidity of the parabolic hoof EFBD.

And $\frac{1}{D-d} \sqrt{4b^2 + (D-d)^2} \times (\text{seg. } FBE - \frac{2}{3} \overline{D-d} \times \sqrt{d \times \overline{D-d}}) =$ convex surface of EFBD.

4. *When the section cuts off part of the base, and makes the angle D r B greater than the angle C A B.*



Let the area of the hyperbolic section EDF $= A$, and the area of the circular seg. EBF $= a$.

Then $\frac{\frac{1}{8} b}{D - b} \times (a \times D - A \times \frac{d \times Br}{cr}) =$ solidity of the hyperbolic ungula EFBD.

And $\frac{1}{D-d} \times \sqrt{4b^2 + (D-d)^2} \times (\text{cir. seg. } EBF - \frac{d^2}{D^2} \times \frac{Br - \frac{1}{2}(D-d)}{Br - D - d} \sqrt{\frac{Br}{Br - D - d}} =$ curve surface of EFBD.

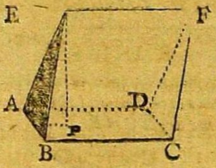
Note. The transverse diameter of the hyp. seg. $= \frac{d \times cr}{D - d - Br}$ and the conjugate $= d \sqrt{\frac{Br}{D - d - Br}}$, from which its area may be found by the former rules.

* *Demon.* When the length of the base is equal to that of the edge, the wedge is evidently equal to half a prism of the same base and altitude. And

height of the wedge and the breadth of the bafe; and $\frac{1}{6}$ of the laft product will be the folidity.

EXAMPLES.

1. How many folid feet are there in a wedge, whofe bafe is 5 feet 4 inches long, and 9 inches broad, the length of the edge being 3 feet 6 inches, and the perpendicular height 2 feet 4 inches?



Here $\frac{(64\times2+42)\times28\times9}{6}=\frac{(128+42)\times28\times9}{6}=\frac{170\times28\times9}{6}=\frac{170\times28\times3}{2}=170\times14\times3=7140$ *folid inches.*

And according as the edge is fhorter or longer than the bafe, the wedge is greater or lefs than half a prifm, by a pyramid of the fame height and breadth at the bafe with the wedge, and the length of whofe bafe is equal to the difference of the lengths of the edge and bafe of the wedge.

Therefore, let the length of the bafe BC $=$ L; the length of the edge EF $=l$; the breadth of the bafe BA $=b$; and the height of the wedge EP $=h$; and we fhall have by the former rules $\frac{blh}{2}\pm bh\times\frac{\pm L\mp l}{3}=\frac{blh}{2}+bh\times\frac{L-l}{3}=bh\times\frac{3l+2L-2l}{6}=bh\times\frac{2L+l}{6}$. Q.E.D.

And

And 7140 ÷ 1728 = 4.1319 *solid feet, the content required.*

2. The length and breadth of the baſe of a wedge are 35 and 15 inches, and the length of the edge is 55 inches: what is the ſolidity, ſuppoſing the perpendicular height to be 17.14508 inches?

Anſ. 3.1006 *feet.*

PROBLEM XI.

To find the ſolidity of a priſmoid.

RULE.*

To the ſum of the areas of the two ends add four times the area of a ſection parallel to and equally diſtant from both ends, and this laſt ſum multiplied by $\frac{1}{6}$ of the height will give the ſolidity.

Note. The length of the middle rectangle is equal to half the ſum of the lengths of the rectangles of the two ends, and its breadth equal to half the ſum of the breadths of thoſe rectangles.

EXAM-

---

* *Demon.* The rectangular priſmoid is evidently compoſed of two wedges, whoſe heights are equal to the height of the priſmoid, and their baſes its two ends. Wherefore, by the laſt problem, its ſolidity will be $= (\overline{2L + l} \times B + \overline{2l + L} \times b) \times \frac{b}{6}$, which, by putting $M = \frac{L + l}{2}$, and $m = \frac{B + b}{2}$, becomes $\overline{BL + bl + 4Mm} \times \frac{b}{6}$; which is the rule, as was to be ſhewn.

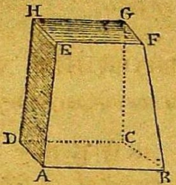
The ſolidities of the two parts, commonly called the ungulas, or hoofs, into which the fruſtum of a rectangular pyramid is divided, may be found by the two laſt rules, as they are only compoſed of wedges and priſmoids.

A very elegant demonſtration of this rule for the priſmoid may be ſeen in *Simpſon*'s *Fluxions, page* 179, 2*d Edition.*

If

EXAMPLES.

1. What is the ſolidity of a rectangular priſmoid the length and breadth of one end being 14 and 12 inches, and the correſponding ſides of the other 6 and 4 inches; and the perpendicular $30\frac{1}{2}$ feet?



Here $14 \times 12 + 6 \times 4 = 168 + 24 = 192$ = *ſum of the areas of the two ends.*

Alſo $\frac{14+6}{2} = \frac{20}{2} = 10$ = *length of the middle rectangle.*

And $\frac{12+4}{2} = \frac{16}{2} = 8$ = *breadth of the middle rectangle.*

Whence $10 \times 8 \times 4 = 80 \times 4 = 320$ = 4 *times the area of the middle rectangle.*

Or $(320 + 192) \times \frac{366}{6} = 512 \times 61 = 31232$ *ſolid inches.*

And $31242 \div 1728 = 18.074$ *ſolid feet, the content.*

2. What is the ſolid content of a priſmoid, whoſe greater end meaſures 12 inches by 8, the leſſer end 8 inches by 6, and the length, or height, 60 inches?

Anſ. 2.453 *feet.*

If the baſes of the priſmoid are diſſimilar rectangles, of which L, *l* and M, *m* are correſponding dimenſions, and *h* the height; Then $(2L + l.\ M + 2l + L.m) \times \frac{1}{6} h$ = ſolidity.

3. What

3. What is the capacity of a coal waggon, whoſe inſide dimenſions are as follows: at the top, the length is $81\frac{1}{2}$, and breadth 55 inches; at the bottom, the length is 41, and the breadth $29\frac{1}{2}$ inches; and the perpendicular depth is $47\frac{1}{4}$ inches?

Anſ. 126340.59375 *cubic inches; which is nearly equal to a chaldron of coals.*

PROBLEM XII.

To find the convex ſurface of a ſphere.

RULE.*

Multiply the diameter of the ſphere by its circumference, and the product will be the convex ſuperficies required.

Note. The curve ſurface of any zone or ſegment will alſo be found by multiplying its height by the whole circumference of the ſphere.

EXAM-

---

* *Demon.* Put the diameter BC $= d$, BA $= x$, AC $= y$, BC $= z$; and 3.14159, &c. $= p$.

Then, ſince the triangles AOC and CED are ſimilar, we ſhall have CA (y) : CO $(\frac{d}{2})$ :: CE $(\dot{x})$: CD $(\dot{z}) = \frac{d\dot{x}}{2y}$. But $2py\dot{z}$ is the general expreſſion for the fluxion of any ſurface; and therefore by ſubſtituting $\frac{d\dot{x}}{2y}$ for its equal $\dot{z}$, the fluxion will become $pd\dot{x}$; and conſequently $pdx =$ ſurface of any ſegment of a ſphere whoſe height is x, and $pdd =$ that of the whole ſphere. Q. E. D.

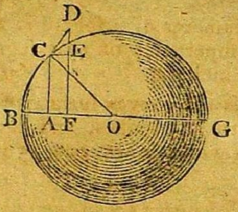
Cor. 1. The ſurface of the ſphere is equal to the curve ſurface of its circumſcribing cylinder.

Cor. 2. The ſurface of a ſphere is alſo equal to 4 times the area of a great circle of it.

1. To

EXAMPLES.

1. What is the convex ſuperficies of a globe B C G whoſe diameter B G is 17 inches?



Here $3.14159 \times 17 \times 17 = 53.40703 \times 17 = 907.91951$ *ſquare inches.*

And, $907.91951 \div 144 = 6.30499$ *ſquare feet the anſwer.*

---

1. To find the lunar ſurface included between two great circles of the ſphere.

RULE. Multiply the diameter into the breadth of the ſurface in the middle, and the product will be the ſuperficies required.

OR, As one right angle is to a great circle of the ſphere;
So is the angle made by the two great circles,
To the ſurface included by them.

2. To find the area of a ſpherical triangle, or the ſurface included by the interſecting arcs of three great circles of the ſphere.

RULE. As two right angles, or 180°,
Is to a great circle of the ſphere;
So is the exceſs of the three angles above two right angles,
To the area of the triangle.

2. What

2. What is the convex superficies of a sphere whose diameter is $1\frac{1}{3}$ feet, and the circumference 4.1888 feet? *Ans.* 5.58506 *feet.*

3. If the diameter, or axis of the earth, be $7957\frac{3}{4}$ miles; what is the whole surface, supposing it to be a perfect sphere? *Ans.* 198943653 *square miles.*

4. The diameter of a sphere is 21 inches; what is the convex superficies of that segment of it whose height is $4\frac{1}{2}$ inches? *Ans.* 296.8802 *inches.*

5. What is the convex surface of a spherical zone, whose breadth is 4 inches, and the diameter of the sphere, from which it was cut, 25 inches?

Ans. 314.16 *inches.*

PROBLEM XIII.

To find the solidity of a sphere or globe.

RULE.*

Multiply the cube of the diameter by .5236, and the product will be the solidity.

EXAM-

---

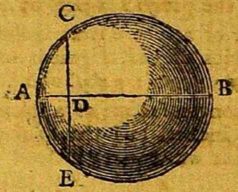
* *Demon.* Put AD $= x$, CD $= y$, the diameter AB $= d$, and $p = 3.14159$.

Then, by the property of the circle, $dx - x^2 = y^2$. But the general expression for the fluxion of any solid is $py^2\dot{x}$; and therefore by writing $dx - x^2$ for its equal y^2, we shall have $p\dot{x} \times \overline{dx - x^2} = pdx\dot{x} - px^2\dot{x}$. The fluent of which is $\frac{pdx^2}{2} - \frac{px^3}{3} = \frac{3pdx^2 - 2px^3}{6} =$ content of the segment C B F.

 And

EXAMPLES.

1. What is the solidity of the sphere A E B C, whose diameter A B is 17 inches?



Here $17^3 \times .5236 = 17 \times 17 \times 17 \times .5236 = 289 \times 17 \times .5236 = 4913 \times .5236 = 5272.4468$ *solid inches*;

And $5272.4468 \div 1728 = 1.48868$ *solid feet the answer.*

2. What is the solidity of a sphere whose diameter is $1\frac{1}{3}$ feet? *Ans.* 1.2411 *feet.*

3. What is the solidity of the earth, supposing it to be perfectly spherical, and its diameter $7957\frac{4}{5}$ miles? *Ans.* 263858149120 *miles.*

PROBLEM XIV.

To find the solidity of the segment of a sphere.

And if d be substituted for x it will become $\frac{3pd^3 - 2pd^3}{9} = \frac{pd^3}{6} = d^3 \times .5236$, or $.5236d^3$; which is the same as the rule.

Coroll. A sphere, or globe, is equal to two-thirds of its circumscribing cylinder.

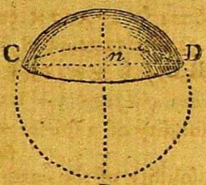
RULE.

RULE.*

To three times the ſquare of the radius of its baſe add the ſquare of its height; and this ſum multiplied by the height, and the product again by .5236 will give the ſolidity.

EXAMPLES.

1. The radius Cn of the baſe of the ſegment CAD is 7 inches, and the height An 4 inches: what is its ſolidity?



* *Demon.* Let r = radius of the baſe of the ſegment, h = height of the ſegment, and the other letters as before.

Then will $(3dh^2 - 2h^3) \times \frac{p}{6}$ = ſolidity of the ſegment, as is ſhewn in the laſt problem.

But ſince $\frac{r^2 + h^2}{h} = d$, by the property of the circle, we ſhall have $\left(\frac{3r^2h^2 + 3h^4}{h} - 2h^3\right) \times \frac{p}{6} = \overline{3r^2 + h^2} \times \frac{ph}{6}$ = ſolidity of the ſegment, which is the ſame as the rule.

Or if d = diameter of the ſphere, and h = height of the ſegment; then will $.5236h^2 \times (3d - 2h)$ = ſolidity.

Here $(7^2\times3+4^2)\times4\times.5236=(49\times3+4^2)\times4\times.5236=(147+4^2)\times4\times.5236=(147+16)\times4\times.5236=163\times4\times.5236=652\times.5236=341.3872$ *solid inches, the anſwer.*

2. What is the ſolidity of the ſegment of a ſphere, the diameter of whoſe baſe is 20, and its height 9?

Anſ. 1799.6132.

3. What is the content of the ſpherical ſegment, whoſe height is 4 inches, and the radius of its baſe 8?

Anſ. 435.6352.

4. What is the ſolidity of a ſpherical ſegment, the diameter of its baſe being 17.23368, and its height 4.5?

Anſ. 572.5566.

PROBLEM XVI.

To find the ſolidity of a fruſtum or zone of a ſphere.

RULE.*

To the ſum of the ſquares of the radii of the two ends, add $\frac{1}{3}$ of the ſquare of their diſtance, or the

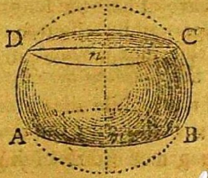
* *Demon.* The difference between two ſegments of a ſphere whoſe heights are H and b, and the radii of whoſe baſes are R and r, will, by the laſt problem $=\frac{p}{6}\times(3R^2H+H^3-3r^2b-b^3)=$ zone whoſe height is $H-b$. And therefore by putting a for the altitude of the fruſtum, and exterminating H and b by the means of the two equations $\frac{3R^2+H^2}{H}=\frac{r^2+b^2}{b}$ and $H-b=a$, we ſhall have $(R^2+r^2+\frac{a^2}{3})\times\frac{pa}{2}$, which is the rule.

breadth

breadth of the zone, and this sum multiplied by the said breadth, and the product again by 1.5708 will give the solidity.

EXAMPLES.

1. What is the solid content of the zone ABCD, whose greater diameter AB is 20 inches, the lesser diameter CD 15 inches, and the distance *nm* of the two ends 10 inches?



Here $(10^2 + 7.5^2 + \frac{10^2}{3}) \times 10 \times 1.5708 = (100 + 56.25 + 33.33) \times 10 \times 1.5708 = 189.58 \times 10 \times 1.5708 = 1895.8 \times 1.5708 = 2977.92264$ *solid inches, the answer.*

2. What is the solid content of a zone, whose greater diameter is 24 inches, the lesser diameter 20 inches, and the distance of the ends 4 inches?

Ans. 1566.6112 *inches.*

3. Required the solidity of the middle zone of a sphere, whose top and bottom diameters are each 3 feet, and the breadth of the zone 4 feet.

Ans. 61.7848 *feet.*

If it be the middle zone of the sphere, the solidity will be $= (d^2 + \frac{2}{3}b^2) \times .7854b$; where $d =$ diameter of each end, and $b =$ its height.

PROBLEM XVII.

To find the surface of a circular spindle, the length and breadth, or middle diameter, being given.

RULE.*

1. To the square of half the length of the spindle, or longest diameter, add the square of half the middle diameter, and this sum divided by the middle diameter will give the *radius of the circle.*

2. Take half the middle diameter from the radius of the circle, and it will give the *central distance.*

3. Find the length of the revolving arc by problem the 10th.

4. From the product of the longest diameter and the radius of the revolving arc, subtract the product of the said arc and the central distance, and this remainder multiplied by 6.2832 will give the surface required.

EXAM-

---

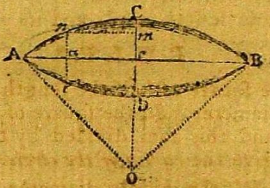
* *Demon.* Put $z =$ arc cn, $x =$ its sine nm, $r =$ radius c o $c =$ central distance oe, and $p = 3.14159$, &c.

Then $\sqrt{r^2 - x^2}\ (om) : r :: \dot{x} : \dot{z}$, and $\dot{z}\sqrt{r^2 - x^2} = r\dot{x}$, by the property of the circle, as is shewn in the demonstration to problem 13. But $2p\dot{z} \times na = 2p\dot{z}(\sqrt{r^2 - x^2} - c) = 2p \times (\dot{z}\sqrt{r^2 - x^2} - c\dot{z})$ is the general expression for the fluxion, and therefore $2p \times (r\dot{x} - c\dot{z}) =$ fluxion of half the frustum cn:p, and $2p \times (rx - cz) = 2p \times (r \times ae - c \times$ c$n) =$ to its surface. Q. E. D.

Coroll.

EXAMPLES.

1. What is the superficial content of the circular spindle ADBC, whose length AB is 48, and the middle diameter CD 36?



Here $(24^2+18^2)\div 36=(576+324)\div 36=\frac{900}{36}=\frac{100}{4}=25=$ oc *the radius of the circle* ACB.

Therefore $25-18=7=$ oc $-$ ce $=$ *central distance* oe.

And $\frac{2\times 18}{3}\div\left(50-\frac{41\times 18}{50}\right)=12\div\overline{50-14.76}=12\div 35.24=.34052.$

Coroll. 1. When $ea=eA$, the rule becomes $2p\times(r\times eA-c\times cA)$ for the surface of half the spindle, and $2p\times\overline{r\times AB-c\times ACB}$ for that of the whole spindle.

Coroll. 2. If from the surface of the semi-spindle there be taken that of the frustum, there will remain $2p\times\overline{r\times aA-c\times An}$ for the segment nAc; so that the rule is general.

Coroll. 3. When e coincides with o, c vanishes, and the spindle becomes a sphere; in which case the theorem also becomes barely $2prx$, the same expression as was before found for the surface of the sphere.

Whence $(1+.34052)\times 48 = 1.34052\times 48 = 64.34496$ = *length of the arc* ACB.

And $(48\times 25 - 64.34496\times 7)\times 6.2832 = (1200 - 450.41472)\times 6.2832 = 749.58528\times 6.2832 = 4709.794231296$ = *superficies required.*

2. What is the superficial content of a circular spindle whose length is 48, and its middle diameter 30? *Ans.* 4387.1644.

PROBLEM XVIII.

To find the solidity of a circular spindle, the length and middle diameter being given.

RULE.*

1. Find the area of the generating circular segment by problem the 13th; and the radius and central distance as in the last problem.

2. From $\frac{1}{3}$ of the cube of half the length of the spindle subtract the product of the central distance and

---

* *Demon.* Put $ae = x$, $oe = c$, r = radius, and $p = 3.14159$, &c.

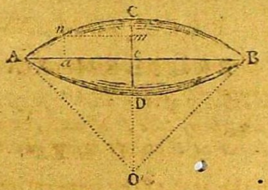
Then will the fluxion of the solid be $= p\dot{x}\times \overline{na}^2 = p\dot{x}\times \overline{(mo-c)}^2 = p\dot{x}\times \overline{mo^2 - c\times 2mo - c} = p\dot{x}\times \overline{mo^2 - c\times 2an + c} = p\dot{x}\times (r^2 - x^2 - c^2 - 2c\times an)$; whose fluent is $= px\times (r^2 - \frac{x^2}{3} - c^2 - \frac{2c}{x}\times ance) = p\times (\overline{Ae^2 - \frac{1}{2}x^2}\times x - 2c\times ance) =$ frustum generated by $ance$. And when $x = AE$, we have $\overline{\frac{1}{3}AE^3 - c\times ACE}\times 2p$ for the half spindle CAD, and $\overline{\frac{1}{3}AE^3 - c\times ACE}\times 4p$ = whole spindle ADBC. Q. E. D.

Corol.

and half the area of the generating ſegment, and this remainder multiplied by 12.5664 will give the ſolidity.

EXAMPLES.

1. The longeſt diameter A B of the circular ſpindle A D B C is 48, and the middle diameter C D 36: what is the ſolidity of the ſpindle?



Here $\frac{48 \times 7}{2} = 24 \times 7$ (*7 being the central diſtance* O *e*, *as in the laſt prob.*) $= 168 =$ *area of the triangle* A C B.

And 32.17248 ($\frac{1}{2}$ *length of the arc* A C B *by laſt prob.*) $\times$ 25 (*radius* O C *by ſame prob.*) $= 804.312 =$ *area of the ſector* B C A O.

Coroll. 1. If the fruſtum be taken from the half ſpindle there will remain $p \times (\overline{Ae}^2 \times Aa - \frac{1}{3} \times \overline{Ae^3 - x^3} - 2c \times An \cdot a) = p \times (\frac{1}{3} Aa^3 \times \overline{3Ae - Aa} - 2c \times An \cdot a)$ for the ſegment of the ſpindle twice $n^1 A\, a$.

Coroll. 2. When *e* coincides with O, *c* will vaniſh, and the theorem will become $\frac{4}{3} p \times OC^3 = \frac{1}{6} p \times D^3$, which is the ſolidity of the whole ſphere.

Which theorems agree with thoſe before given.

Alſo $804.312-168=636.312=$ *area of the ſegment ACBe.*

And $(\frac{24^3}{3}-\frac{636.312}{2}\times 7)\times 12.5664=(\frac{13824}{3}-318.156\times 7)\times 12.5664=(4608-2227.092)\times 12.5664=2380.908\times 12.5664=29919.4422912=$ *ſolidity required.*

2. If the length of a circular ſpindle be 40, and its middle diameter 30: what is its ſolidity?

Anſ. 17310.458.

PROBLEM XIX.

To find the ſolidity of the middle fruſtum of a circular ſpindle, the length of the fruſtum, the middle diameter, and that of either of the ends being given.

RULE.*

1. Divide the ſquare of half the length of the fruſtum by half the difference of the middle diameter, and that of either of the two ends; and half this quotient added to $\frac{1}{4}$ of the ſaid difference will give the radius of the circle.

2. Find the central diſtance, and the revolving area, as in the laſt problem.

3. From the ſquare of the radius take the ſquare of the central diſtance, and the ſquare root of the remainder will give half the length of the ſpindle.

---

* The demonſtration is contained in that of the laſt problem.

The ſolidity of the ſegment cannot be found independant of the length of the ſpindle. The rule may be ſeen in the Corollary to the laſt problem.

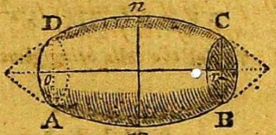
4. From

4. From the ſquare of half the length of the ſpindle take $\frac{1}{3}$ of the ſquare of half the length of the fruſtum, and multiply the remainder into the ſaid half length.

5. From this product take that of the generating area and central diſtance, and the remainder multiplied by 6.2832 will give the content of the fruſtum.

EXAMPLES.

1. What is the ſolidity of the fruſtum ABCD, whoſe middle diameter *nm* is 36, the diameter DA or CB 16, and the length *or* 40?



Here $\frac{1}{2} \times (20^2 \div \frac{36-16}{2}) + \frac{36-16}{4} = \frac{1}{2} \times (400 \div 10) + 5 = \frac{1}{2} \times 40 + 5 = 20 + 5 = 25 =$ *radius of the circle.*

Conſequently $25 - \frac{1}{2} nm = 25 - 18 = 7 =$ *central diſtance.*

And $\frac{10}{50} = \frac{1}{5} = .2 =$ *tab. verſed ſine; and* $.111823 =$ *tab. ſegment.*

Alſo $.111823 \times 50^2 = .111823 \times 2500 = 279.5575 =$ *area of the ſegment* DNC.

And $279.5575 + 320 = 599.5575 =$ *generating area* ODNCr.

 Again

Again $\sqrt{(25^2-7^2)} = \sqrt{(625-49)} = \sqrt{576} = 24$ $= \frac{1}{2}$ *length of the ſpindle.*

And $(24^2 - \frac{20^2}{3}) \times 20 - 599.5575 \times 7) \times 6.2832 =$ $(556 - 133.3) \times 20 - 4196.9025) \times 6.2832 =$ $(8853.334 - 4196.9025) \times 6.2832 = 4656.4135 \times$ $6.2832 = 29257.2904$ *ſolidity required.*

2. The middle diameter of the fruſtum of a circular ſpindle is 32, the diameter at the end 24, and the length 40; what is the ſolidity?

Anſ. 27287.5411256 *cubic inches.*

PROBLEM XX.

To find the ſolid of a ſpheroid.

RULE.*

Multiply the ſquare of the revolving axe by the fixed axe, and this product again by .5236, and it will give the ſolidity required.

Where note that .5236 is $= \frac{1}{6}$ of 3.14159.

EXAM-

---

* *Demon.* Let AC $= a$, DB $= b$, Ar $= x$, $rn = y$, and $p = 3.14159$, &c.

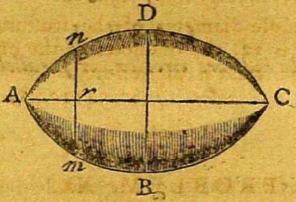
Then $a^2 : b^2 :: x \times (a-x) : \frac{b^2}{a^2} \times (ax - x^2) = y^2$ by the property of the ellipſis.

And therefore the fluxion of the ſolid $(= py^2\dot{x}) = \frac{pb^2}{a^2} \times$ $(ax\dot{x} - x^2\dot{x})$; and its fluent $= \frac{pb^2}{a^2} \times (\frac{1}{2}ax^2 - \frac{1}{3}x^3) =$ ſegment

nAm,

EXAMPLES.

1. In the prolate ſpheroid ABCD, the tranſverſe or fixed axe AC is 90, and the conjugate or revolving axe DB is 70: what is the ſolidity?



Here $DB^2 \times AC \times .5237 = 70^2 \times 90 \times .5236 = 4900 \times 90 \times .5236 = 441000 \times .5236 = 230907.6 =$ *ſolidity required.*

2. What is the ſolidity of a prolate ſpheroid, whoſe fixed axe is 100, and its revolving axe 60?

Anſ. 188496.

3. What is the ſolidity of an oblate ſpheroid, whoſe fixed axe is 60, and its revolving axe 100?

Anſ. 314160.

---

nAm. Which, when $x = a$, becomes $\frac{pb^2}{a^2} \times (\frac{1}{2}a^3 - \frac{1}{3}a^3) = \frac{pab^2}{6} =$ content of the whole ſpheroid. Q. E. D.

If f be put = fixed axe, r = revolving axe, $q = (f^2 \backsim r^2) \div f^2$, and $p = 3.1415$, &c.

Then will $prf \sqrt{1 + \frac{1}{3}q}$ = ſurface of the oblate ſpheroid, and $prf \sqrt{1 - \frac{1}{3}q}$ = that of the prolate ſpheroid.

PRO-

PROBLEM XXI.

To find the content of the middle frustum of a spheroid, its length, the middle diameter, and that of either of the ends being given.

CASE I.

When the ends are circular, or parallel to the revolving axis.

RULE.*

To twice the square of the middle diameter add the square of the diameter of either of the ends, and this sum multiplied by the length of the frustum, and the product again by .2618, will give the solidity.

Where note that $.2618 = \frac{1}{12}$ of 3.14159.

EXAM.

---

* *Demon.* Let AO $= a$, DO $= b$, EN $= b'$, NO $= c$, RO $= x$, RE $= y$, and $p = 3.14159$, &c.

Then $a : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times (a^2 - x^2) = b^2 - \frac{b^2x^2}{a^2} =$ y^2 by the property of the ellipsis.

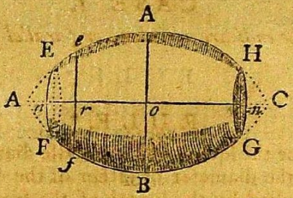
And also $a^2 : b^2 :: \overline{a^2 - c^2} : \frac{b^2 \times (a^2 - c^2)}{a^2} = b'^2$; or $a^2 = b^2c^2 \div (b^2 - b^2)$.

Whence, by substituting this value of a^2 in the former equation, we shall have $y^2 = b^2 - \frac{b^4x^2 - b^2b'^2x^2}{b^2c^2} = b^2 - \frac{b^2x^2 - b^2x^2}{c^2} = b^2 - \frac{x^2}{c^2} \times (b^2 - b^2)$.

And

EXAMPLES.

1. In the middle frustum of a spheroid E F G H, the middle diameter D B is 50 inches, and that of either of the ends E F or G H 40 inches, and its length *n m* 18 inches: what is its solidity?



Here $(50^2 \times 2 + 40^2) \times 18 \times .2618 = (2500 \times 2 + 1600) \times 18 \times .2618 = (5000 + 1600) \times 18 \times .2618 = 6600 \times 18 \times .2618 = 118800 \times .2618 = 31101.84$ *cubic inches, the answer.*

2. What is the solidity of the middle frustum of a prolate spheroid, the middle diameter being 60, that of either of the two ends 36, and the distance of the ends 80? *Ans.* 177940.224.

And consequently the fluxion of the solid $(py^2\dot{x}) = pb^2\dot{x} - \frac{px^2\dot{x}}{c^2} \times (b^2 - b^2)$; the fluent of which is $= pb^2x - \frac{px^3}{3c^2} \times (b^2 - b^2)$; which, when $x = c$, becomes $pb^2c - \frac{pcb^2 - pcb^2}{3} = \frac{pc \times (2b^2 + b^2)}{3} = \frac{pc}{12} \times \overline{8b^2 + 4b^2}$. Q. E. D.

3. What

3. What is the ſolidity of the middle fruſtum of an oblate ſpheroid, the middle diameter being 100, that of either of the ends 80, and the diſtance of the ends 36?

Anſ. 248814.72.

CASE II.

When the ends are elliptical or perpendicular to the revolving axis.

RULE.*

1. Multiply twice the tranſverſe diameter of the middle ſection by its conjugate diameter, and to this product add the product of the tranſverſe and conjugate diameters of either of the ends.

2. Mul-

---

* *Demon.* Put SO $= a$, EO $= b$, $om = r$, $on = x$, A$n = y$, nC $= z$, and $p = 3.14159$, &c.

Then $a^2 : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times (a^2 - x^2) = y^2$ by the property of the ellipſis.

And, ſince ACD is an ellipſis ſimilar to EmF, it will be $b : r :: y : \frac{ry}{b} = z$; as is ſhewn by the writers on Conics.

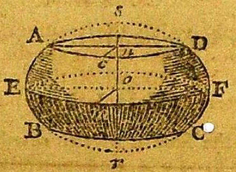
But the fluxion of the ſolid AEFD is $pyz\dot{x} = py\dot{x} \times \frac{ry}{b} = \frac{pry^2\dot{x}}{b} = \frac{pr\dot{x}}{b} \times \frac{b^2 \times (a^2 - x^2)}{a^2} = prb\dot{x} \times \frac{a^2 - x^2}{a^2}$. And the fluent $= prbx \times \frac{a^2 - \frac{1}{3}x^2}{a^2}$. Which, by ſubſtituting for a^2 its value $\frac{b^2 x^2}{b^2 - y^2}$, becomes $= prx \times \frac{2b^2 + y^2}{3b} = px \times \frac{2}{3}rb + \frac{ry^2}{3b}$.

And

2. Multiply the sum, thus found, by the distance of the ends, or the height of the frustum, and the product again by .2618, and it will give the solidity required.

EXAMPLES.

1. In the middle frustum ABCD of an oblate spheroid, the diameters of the middle section EF are 50 and 30; those of the end AD 40 and 24; and its height *ne* 18; what is the solidity?



Here $(50 \times 2 \times 30 + 40 \times 24) \times 18 \times .2618 = (3000 + 960) \times 18 \times .2618 = 3960 \times 18 \times .2618 = 71280 \times .2618 = 18661.104 =$ *solidity required.*

2. In the middle frustum of a prolate spheroid, the diameters of the middle section are 100 and 60; those of the end 80 and 48; and the length 36: what is the solidity? *Ans.* 149288.832.

And this again, by putting z for its equal $\frac{ry}{b}$ becomes $= \frac{px}{3} \times \overline{2rb + yz} =$ frustum EFDA. Or $\frac{p \times ne}{12} \times (2EF \times 2om + AD \times 2nc) =$ middle frustum ABCD. Q.E.D.

3. In the middle fruſtum of an oblate ſpheroid, the diameters of the middle ſection are 100 and 60; thoſe of the end 60 and 36; and the length 80: what is the ſolidity of the fruſtum? *Anſ.* 296567.04.

PROBLEM XXII.

To find the ſolidity of the ſegment of a ſpheroid.

CASE I.

When the baſe is parallel to the revolving axis.

RULE.*

1. Divide the ſquare of the revolving axis by the ſquare of the fixed axe, and multiply the quotient by the difference between three times the fixed axe and twice the height of the ſegment.

2. Multiply the product, thus found, by the ſquare of the height of the ſegment, and this product again by .5236, and it will give the ſolidity required.

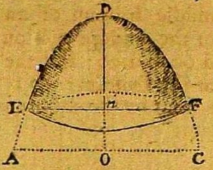
EXAMPLES.

1. In the prolate ſpheroid DEFD, the tranſverſe axis 2DO is 100, the conjugate AC 60, and the height D*n* of the ſegment EDF 10: what is the ſolidity?

---

* This rule is formed from the theorem for the ſegment in the demonſtration to problem the 20th.

60



Here $(\frac{60^2}{100^2} \times \overline{300-10}) \times 10^2 \times .5236 = .36 \times 280 \times 10^2 \times .5236 = 100.80 \times 100 + .5236 = 10080 \times .5236 = 5277.888 =$ *solidity required.*

2. The axes of a prolate ſpheroid are 50 and 30: what is the ſolidity of that ſegment whoſe height is 5, and its baſe perpendicular to the fixed axe?

Anſ. 659.746.

3. The diameters of an oblate ſpheroid are 100 and 60; what is the ſolidity of that ſegment whoſe height is 12, and its baſe perpendicular to the conjugate axe? *Anſ.* 32672.64.

CASE II.

When the baſe is perpendicular to the revolving axis.

The content of the ſegment may alſo be found by the following theorem:

$(D^2 + 4d^2) \times \frac{1}{6} n b =$ content of the ſegment; D being the diameter of the baſe, $d =$ diameter in the middle, $b =$ height, and $n = .7854 =$ area of a circle whoſe diameter is 1.

RULE.

RULE.*

1. Divide the fixed axe by the revolving axe, and multiply the quotient by the difference between three times the revolving axe and twice the height of the segment.

2. Multiply the product, thus found, by the square of the height of the segment, and this product again by .5236, and it will give the solidity required.

EXAMPLES.

1. In the oblate spheroid a E b F, the transverse axe E F is 100, the conjugate ab 60, and the height an, of the segment a A D, 12: what is the solidity?

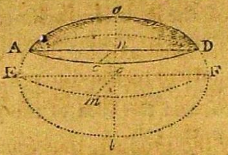
---

* *Demon.* Put $ao = a$, E$o = b$, $om = r$, $an = x$, A$n = y$, $nc = z$, and $p = 3.14159$, &c. Then will $a^2 : b^2 :: a^2 - (a-x)^2$ or $(2ax - x^2) : \frac{b^2 \times (2ax - x^2)}{a^2} = y^2$ by the property of the ellipse.

And, since A C D is an ellipse similar to E m F, it will be $b : r :: y : \frac{ry}{b} = z$; as is shewn by the writers on Conics.

But the fluxion of the solid a A C D $= pyz\dot{x} = py\dot{x} \times \frac{ry}{b} = \frac{pry^2\dot{x}}{b} = \frac{pr\dot{x}}{b} \times \frac{b^2 \times (2ax - x2)}{a^2}$; whose fluent is $= \frac{prb}{a}x^2 - \frac{prb}{3a^2}x^3$; which, when $x = b =$ the height of the segment, becomes $(3ab^2 - b^3) \times \frac{prb}{3a^2}$. Whence, since $r = a$, we shall have $(3ab^2 - b^3) \times \frac{pb}{3a} =$ solidity of the segment.

Q. E. D.



Here 156 (=*dif. of* 3*ab and* 2*an*) × $1\frac{2}{3}$ (= EF, *or* $\frac{100}{ab}$) × 144 (=*square of an*) × .5236 = $\frac{156 \times 5}{3}$ × 144 × .5236 = 52 × 5 × 144 × .5236 = 260 × 144 × .5236 = 37440 × .5236 = 19603.584 = *solidity required.*

2. Required the content of the ſegment of a prolate ſpheroid; its height being 6, and the axes 40 and 24. *Anſ.* 2450.44226.

PROBLEM XXIII.

To find the ſolidity of an elliptic ſpindle.

RULE.*

1. From three times the ſquare of the middle diameter take 4 times the ſquare of the diameter between the middle and the end; and from 4 times this

---

* *Demon.* Let $Fv = a$, $ov = b$, $\frac{1}{2}$ the tranſverſe diameter $= c$, $Do = d$, $va = x$, and vc or $an = y$.

Then, by the property of the ellipſe, $c : d :: \sqrt{c^2 - x^2} : \frac{d}{c}\sqrt{c^2 - x^2} = co$; hence cv or $an = co - ov = \frac{d\sqrt{c^2 - x^2}}{c} - b = y$, and the fluxion of the ſolid $= py^2\dot{x} = p\dot{x} \times$ (a^2

this laſt diameter take 3 times the ſaid middle diameter; and $\frac{1}{4}$ of the quotient ariſing from dividing the former difference by the latter will give the *central diſtance*.

2. Find the axes of the ellipſis by problem the 2d, in conic ſections, and the area of the generating ſegment by problem the 5th.

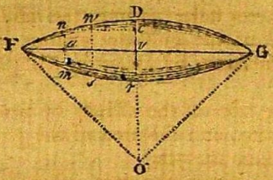
3. Divide 3 times the area, thus found, by the length of the ſpindle, and from the quotient ſubtract the middle diameter; then multiply the remainder by 4 times the central diſtance, and ſubtract the product from the ſquare of the middle diameter; and this difference multiplied by $\frac{1}{3}$ of the length of the ſpindle, and the product again by 1.57079 will give the ſolidity.

EXAMPLES.

1. What is the ſoldity of the elliptic ſpindle FrGD, whoſe length FG is 80, the middle diameter Dr 24, and the diameter ws at $\frac{1}{4}$ of the length 18.99094?

Here

$(d^2 - \frac{d^2x^2}{c} - \frac{2bd\sqrt{c^2-x^2}}{c} + b^2) = p\dot{x} \times (d^2 - b^2 - \frac{d^2x^2}{c^2} - \frac{2bd\sqrt{c^2-x^2}}{c}) - b = pd^2\dot{x} \times \frac{a^2-x^2}{c^2} - 2pby\dot{x}$; and its fluent $= pd^2x \times \frac{3a^2-x^2}{3c^2} - 2bp \times$ area $vanD =$ fruſtum $Dnmr$. And when $x = a$ the above theorem will become $\frac{2pd^2}{3c^2} \times a^3 - 2bp \times$ area $FDv = DFr$ the half of the ſpindle.



Here $(3 \times \overline{24^2 - 18.99094^2} \times 4 \div \overline{18.99094 \times 4 - 3 \times 4}) \times \frac{1}{4} = (\overline{1728 - 1442.62320833} \div \overline{75.96376 - 72}) \times \frac{1}{4} = (285.37679167 \div 3.96376) \times \frac{1}{4} = 71.996 \times \frac{1}{4} = 17.999 = 18$ *nearly, for the central diſtance* O v.

And $18 + \frac{24}{2} = 18 + 12 = 30 = \frac{1}{2}$ *conjugate* DO. *And* $\frac{30}{2} \times 30 \div \sqrt{30^2 - 18^2} = 40 \times 30 \div \sqrt{900 - 324} = 1200 \div \sqrt{576} = \frac{1200}{24} = 50 = \frac{1}{2}$ *tranſverſe.*

Alſo $\frac{12}{2 \times 30} = \frac{12}{60} = \frac{1}{5} = .2 =$ *tab. verſed ſine; the circular ſegment to which is* $.111823 \therefore .111823 \times 100$ (AB) $\times 60$ (DF) $= 11.1823 \times 60 = 670.938 =$ *generating ſegment* FDG.

24²

ſpindle. And if from the ſemi-ſpindle there be taken the fruſtum, there will remain $pd^2 e^2 \times \frac{3a - e}{3c^2} - 2pb \times$ area $Fna =$ ſegment nFm, e being the height Fa of the ſegment. But to convert theſe rules into thoſe given in the text, let $Dr = D$, $nm = d$, $ws = m$, $va = b$, $ov = c$, the ſegment $nDc = s$, and $n = .7854$.

Then, by the property of the ellipſe, $(c + \frac{1}{2}D)^2 - (c + \frac{1}{2}d) : 4 :: (c + \frac{1}{2}D)^2 - (c + \frac{1}{2}m)^2 : 1$; hence $4 \times (c + \frac{1}{2}m)^2 - 3 \times (c + \frac{1}{2}D)^2 = (c + \frac{1}{2}d)^2$, or $c = \frac{1}{4} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$.

And

Whence $24^2 - [(\frac{670.938 \times 3}{80} - 24) \times 72 \text{ or } 40v] =$
$24^2 - (25.160175 - 24 \times 72) = 24^2 - 1.160175 \times 72 =$
$24^2 - 83.5326 = 576 - 83.5326 = 492.4674.$ *And*
$492.4674 \times \frac{80}{3} \times 1.57079 = 13132.464 \times 1.57079 =$
$20628.34312656 =$ *solidity required.*

2. The length of an elliptic ſpindle is 40, the middle diameter 12, and the diameter at $\frac{1}{4}$ of the length 9.49546: what is the ſolidity? *Anſ.* 2578.56.

PROBLEM XXIV.

To find the ſolidity of the middle fruſtum of an elliptic ſpindle; the length, the diameters of the middle and end, and another parallel thereto at $\frac{1}{4}$ of the length of the fruſtum, being given.

RULE.*

1. From the ſum of 3 times the ſquare of the middle diameter and the ſquare of that of the end

---

And the laſt two theorems become $\frac{1}{3}nh \times (\frac{1}{2}D^2 + d^2 - 2) \times \frac{3D^2 + d^2 - 4m}{-3D - d + 4m} \times (-D + d + \frac{3s}{h})$ for the fruſtum $Dnmr$; and $\frac{1}{3}nl \times (2D^2 - 2 \times \frac{3D^2 - 4m^2}{-3D + 4m} \times -D + \frac{3s}{l})$ for the ſemi-ſpindle DFr, where $l = vF$ and $s =$ area of FDv.

Q. E. D.

* The demonſtration of this rule is contained in that of the laſt problem.

The ſolidity of a ſegment of an elliptic ſpindle cannot be found independently of the axes of the ſpheroid.

take

take 4 times the square of the diameter between the middle and the end; and from 4 times this last diameter take the sum of the least diameter and 3 times that of the middle, and $\frac{1}{4}$ of the quotient arising from dividing the former difference by the latter will give the *central distance*.

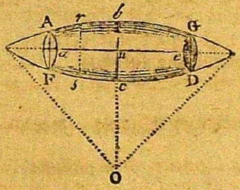
2. Find the axes of the ellipse by problem the 2d, and the area of the elliptical segment whose chord is the length of the frustum by problem the 5th.

3. Divide three times the area thus found by the length of the frustum, and from the quotient subtract the difference between the middle diameter and that of the end, and multiply the remainder by 8 times the central distance.

4. Then from the sum of the square of the least diameter, and twice the square of that in the middle, take the product last found, and this difference multiplied by the length, and the product again by .261799, &c. will give the solidity required.

EXAMPLES.

1. What is the solidity of the frustum AFDG, whose length *ae* is 28, the middle diameter *bc* 24, the diameter AF of the end 21.6, and that *rs* at $\frac{1}{4}$ of the length 23.409?



I

Here

Here $\frac{24^2 \times 3 + (21.6)^2 - 4 \times (23.409)^2}{4 \times 23.409 - (21.6 + 3 \times 24)} = \frac{2194.56 - 2191.925124}{93.63636 - 93.6} = \frac{2194.56 - 2191.925124}{.03636} = \frac{2.634876}{.03030}$ $= 72.46$.

And $\frac{72.46}{4} = 18.11 = 18$ *nearly* = *central diſtance on.*

Alſo $\frac{30\,(=bo) \times 14\,(=an)}{\sqrt{30^2(=bo^2) - 28.8^2\,(=om^2)}} = \frac{420}{\sqrt{900 - 829.44}}$ $= \frac{420}{\sqrt{70.56}} = \frac{420}{8.4} = 50 = \frac{1}{2}$ *tranſverſe diameter.*

And $\frac{1.2\,(=bm)}{60\,(=bp)} = .02 =$ *tab. verſed ſine*; *and* .003748 = *circular ſegment belonging to* .02 ; ∴ .003748 × 100 (= LN) × 60 (= *bp*) = 22.488 = *elliptic ſegment, of which* ABG *is the arc.*

Whence $24^2 \times 2 + (21.6)^2 - \left(\frac{3 \times 22.488}{28} - 2.4\,(=\text{diff. of } bc \text{ and } AF) \times 8 \times 18\right) = 1152 + 466.56 - (\overline{2.4094 - 2.4} \times 144) = 1618.56 - .0094 \times 144 = 1618.56 - 1.3536$ $= 1617.2064$.

And $1617.2064 \times 28\,(=ae) \times .261799 = 45281.779^2 \times .261799 = 11854.7245127808 =$ *ſolidity required.*

2. In the middle fruſtum of an elliptic ſpindle, the middle diameter is 32, the diameter at the end 24, and the diameter at $\frac{1}{4}$ of the length 30.15756, and the length 40: required the ſolidity. *Anſ.* 27419.8219.

PROBLEM XXV.

To find the ſolidity of a parabolic conoid.

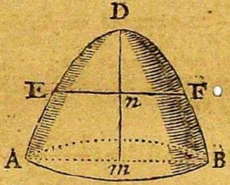
 RULE.

RULE.*

Multiply the area of the base by half the altitude, and the product will be the content.

EXAMPLES.

1. What is the solidity of the paraboloid A D B, whose height D m is 84, and the diameter B A of its circular base 48?



* *Demon.* Let D m = a, B m = b, D n = x, E n = y, and p = 3.14159, &c.

Then by the nature of the parabola $a : b^2 :: x : y^2$, or $\frac{b^2 x}{a} = y^2$; wherefore $\frac{pb^2 x\dot{x}}{a}$ $(= py^2\dot{x})$ = the fluxion of the solid, and $\frac{pb^2 x^2}{2a}$ = its fluent; which, when x becomes = a, is $\frac{1}{2} pab^2$ for the whole solid, or for any segment whose height is = a, and the radius of its base = b. Q. E. D.

Coroll. The parabolic conoid is = $\frac{1}{2}$ its circumscribing cylinder.

Note. The rule given above will hold for any segment of the paraboloid, whether the base be perpendicular or oblique to the axe of the solid.

 2. What

Here $8^2 \times .7854 \times 42 (= \frac{1}{2} nm) = 2304 \times .7854 \times 42 = 1809.5616 \times 42 = 76001.5872 =$ *solidity required.*

2. What is the solidity of a paraboloid, whose height is 60, and the diameter of its circular base 100? *Ans.* 235620.

3. Required the solidity of a parabolic conoid, whose height is 30, and the diameter of its base 40? *Ans.* 18849.6.

4. Required the solidity of a parabolic conoid, whose height is 50, and the diameter of its base 100? *Ans.* 126350.

PROBLEM XXVI.

To find the solidity of the frustum of a paraboloid, when its ends are perpendicular to the axe of the solid.

RULE.*

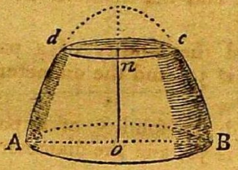
Multiply the sum of the squares of the diameters of the two ends by the height of the frustum, and the product again by .3927, and it will give the solidity.

---

* *Demon.* The segment whose base is B, and altitude A, is $= \frac{1}{2}$ A B, and that whose base is b and altitude a is $= \frac{1}{2} ab$, by the last problem: wherefore the frustum, or the difference of the segments is $\frac{1}{2} AB - \frac{1}{2} ab$. But $B - b : A - a\ (d) ::$ $B : A = \frac{Bd}{B-b}$; and $B - b : d :: b : a = \frac{bd}{B-b}$, by the nature of the paraboloid; and these values of A and a being substituted for them will make $\frac{1}{2} AB - \frac{1}{2} ab = \frac{dB - db}{2B - 2b} = \frac{1}{2} d \times$ $(B + b)$ which is the same as the rule. Q. E. D.

EXAMPLES.

1. Required the solidity of the parabolic frustum A B *c d*; the diameter A B of the greater end being 58, that of the lesser end *dc* 30, and the height *no* 18.



Here $(58^2+30^2)\times18\times.3927=(3364+900)\times18\times.3927=4264\times18\times.3927=76752\times.3927=30140.5104=$ *solidity required.*

2. What is the solidity of the frustum of a parabolic conoid, the diameter of the greater end being 60, that of the lesser end 48, and the distance of the ends 18?

Ans. 41733.0144.

PROBLEM XXVII.

To find the solidity of a parabolic spindle.

RULE.*

Multiply the square of the middle diameter by the length of the spindle, and the product again by .418879, and it will give the solidity.

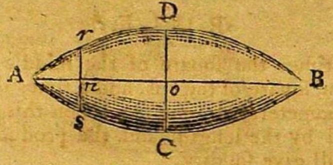
EXAM-

* *Demon.* Put B *o* $=a$, A *o* $=b$, $c=3.14159$, &c. A *n* $=x$, and *r n* $=y$.

 Then

EXAMPLES.

1. The length of the parabolic ſpindle ACBD is 60, and the middle diameter DC 34: what is the ſolidity?



Here $34^2 \times 60 \times .418879 = 1156 \times 60 \times .418879 = 69360 \times .418879 = 29053.44744 =$ *ſolidity required.*

2. The length of a parabolic ſpindle is 9 feet, and the middle diameter 3 feet: what is the ſolidity?

Anſ. 33.929199.

Then it will be, by the property of the parabola, $b^2 : a :: 2bx - x^2$ (A$n \times n$B) $: a \times \frac{2bx - x^2}{b^2} = rn = y$; which, by putting $p = \frac{b^2}{a}$ ($=$ parameter of Do,) becomes $\frac{2bx - x^2}{p}$; and hence the fluxion of the ſolid $= cy^2\dot{x} = \frac{c\dot{x}}{p^2} \times \overline{2bx - x}|^2$; the fluent of which is $cx^3 \times \frac{\frac{4}{3}b^2 - bx + \frac{1}{5}x^2}{p^2} =$ general expreſſion for the ſegment ras; and therefore when $x = b$, we ſhall have $\frac{8cb5}{15p^2} = \frac{8}{15}ca^2b =$ ſemi-ſpindle DAC, and $a^2b \times .418879 =$ whole ſpindle BDAC. Q. E. D.

PRO-

PROBLEM XXVIII.

To find the solidity of the middle fruſtum of a parabolic ſpindle.

RULE.*

Add 8 times the ſquare of the middle diameter, 3 times the ſquare of the leſs, and 4 times the product of thoſe diameters into one ſum; then this ſum being multiplied by the length, and the product again by .05236 will give ſolidity.

EXAMPLES.

1. In the middle fruſtum EFGH, of the parabolic ſpindle ACBD, the middle diameter DC is 36, the

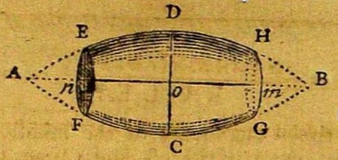
---

* *Demon.* Let $on = a - x = z$, and the other letters as in the laſt problem.

Then if from the value of the ſemi-ſpindle $\frac{8cb^5}{15p^2}$ there be taken $cx^3 \times \frac{\frac{4}{3}b^2 - bx + \frac{1}{5}x^2}{p^2}$ the value of the ſegment EAF, there will remain $\frac{cz}{p^2} \times \overline{b^4 - \frac{2}{3}b^2z^2 + \frac{1}{5}z^4} = \frac{ca^2z}{b^4} \times (b^4 - \frac{2}{3}b^2z^2 \times \frac{1}{5}z^4) =$ the value of the fruſtum DEFC; and if inſtead of z there be ſubſtituted its value $b\sqrt{\frac{a-y}{a}}$ the value of the ſaid fruſtum will be denoted by $c \times no \times \frac{8DC^2 + 4no \times EF + 3EF^2}{60}$, and conſequently $nm \times .05236 \times 8DC^2 + 4no \times EF + 3EF^2 =$ whole fruſtum. Q. E. D.

diameter

diameter of the end E F is 20, and the length *n m* 36: what is the solidity?



Here $(36^2 \times 8 + 20^2 \times 3 + 4 \times 36 \times 20) \times 36 \times .05236 = (10368 + 1200 + 2880) \times 36 \times .05236 = 14448 \times 36 \times .05236 = 520128 \times .0.5236 = 27233.90228 =$ *solidity required.*

2. Required the solidity of the middle frustum of a parabolic spindle, the middle diameter being 32, the diameter at the end 24, and the length 40?

Ans. 17210.448.

PROBLEM XXIX.

To find the solidity of an hyperboloid.

RULE.*

To the square of the radius of the base add the square of the middle diameter between the base and the

---

* *Demon.* Let t = transverse, and c = conjugate diameter of the generating hyperbola, p = 3.14159, y, Y, the ordinates, or semi-diameters of the ends of any frustum of the hyperboloid, x = its altitude, and A = distance of the less ordinate y from the vertex of the whole solid.

Then

the vertex; and this ſum multiplied by the altitude, and the product again by .5236 will give the ſolidity.

Then ſince $\mathrm{Y}^2 = \frac{(t+\mathrm{A}+x)\times(\mathrm{A}+x)}{t^2} \times c^2$, we ſhall have the fluxion of the ſolid $= p\mathrm{Y}^2\dot{x} = pc^2\dot{x} \times \frac{\mathrm{A}t + \mathrm{A}^2 + 2\mathrm{A}x + tx + x^2}{t^2}$, and its fluent $= pc^2x \times \frac{\mathrm{A}t + \mathrm{A}^2 + \mathrm{A}x + \frac{1}{2}tx + \frac{1}{3}x^2}{t^2}$; and this, by ſubſtituting $\frac{y^2}{c^2}$ for $\frac{\mathrm{A}t + \mathrm{A}^2}{t^2}$, and $\frac{\mathrm{Y}^2}{c^2}$ for $\frac{\mathrm{A}t+\mathrm{A}^2+2\mathrm{A}x+tx+x^2}{t^2}$ becomes $(\mathrm{Y}^2 + y^2 - \frac{c^2x^2}{3t^2}) \times \frac{1}{2}px =$ ſolidity of the fruſtum.

But to convert this into the rules given in the text, let D, δ, d, be the greateſt, middle, and leaſt diameters, $x =$ abſciſſa whoſe ordinate is δ, and $a =$ altitude. Then we ſhall have theſe three equations:

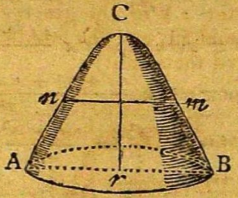
$$t^2\delta^2 = c^2 \times \overline{t+x} \times x$$
$$t^2d^2 = c^2 \times \overline{t+x-\tfrac{1}{2}a} \times \overline{x-\tfrac{1}{2}a}$$
$$t^2\mathrm{D}^2 = c^2 \times \overline{t+x+\tfrac{1}{2}a} \times \overline{x+\tfrac{1}{2}a}$$

From the ſum of the two latter of which ſubtract the double of the former, and there will reſult $t^2 \times \overline{\mathrm{D}^2 - 2\delta^2 + d^2} = \frac{1}{2}a^2c^2$; and hence $\frac{a^2c^2}{3t^2} = \frac{2\mathrm{D}^2 - 4\delta^2 + 2d}{3}$. Which being ſubſtituted for it in the theorem above will give $\frac{\mathrm{D}^2 + 4\delta^2 + d}{6} \times ap$ for the content of the fruſtum; which is the ſame as the following rule given in the text.

And if d the leaſt diameter be ſuppoſed to become infinitely little, or nothing, the rule will become $\frac{\mathrm{D}^2 + 4\delta^2}{6} \times ap = \overline{\mathrm{D}^2 + 4\delta^2} \times a \times .5236$. Q. E. D.

EXAMPLES.

1. In the hyperboloid ACB, the altitude Cr is 10, the radius Ar of the baſe 12, and the middle diameter *nm* 15.8745: what is the ſolidity?



Here $\overline{15.8745^2 + 12^2} \times 10 \times .5236 = \overline{251.99975 + 144} \times 10 \times .5236 = 395.99975 \times 10 \times .5236 = 3959.9975 \times .5236 = 2073.4558691 =$ *ſolidity required.*

2. In an hyperboloid the altitude is 50, the radius of the baſe 52, and the middle diameter 68; what is the ſolidity? *Anſ.* 191847.

PROBLEM XXX.

To find the ſolidity of the fruſtum of an hyperbolic conoid.

RULE.*

Add together the ſquares of the greateſt and leaſt ſemi-diameters, and the ſquare of the whole diameter in

---

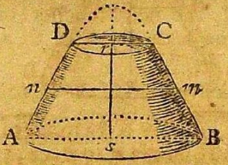
* The demonſtration of this rule is contained in that of the laſt problem.

The

in the middle, then this ſum being multiplied by the altitude, and the product again by .52359 will give the ſolidity.

EXAMPLES.

1. In the hyperbolic fruſtum A D C B, the length *rs* is 20, the diameter A B of the greater end 32, that D C of the leſſer end 24, and the middle diameter *nm* 28.1708: required the ſolidity.



The rule for the hyperbolic ſpindle is the ſame as that for the elliptic ſpindle, page 175.

Or, if D = middle diameter, m = that at $\frac{1}{4}$ of the length, S = generating area of the hyperbola, L = length of the ſpindle, and $p = 3.14159$, &c.

Then will $\left(D + \frac{4m^2 - 3D^2}{4m - 3D} \times \frac{3S - D}{L}\right) \times \frac{1}{6} pL$ = ſolidity of the ſpindle. And if the generating hyperbola be equilateral, then will $\left(3S \times \frac{L^2 + D^2}{LD} - L^2\right) \times \frac{1}{6} pL$ = ſolidity of the ſpindle.

And, if l = length of the fruſtum, s = generating area, and the other letters as before; then will $\left(2D^2 + d^2 + \frac{4m^2 - 3D^2 - d^2}{4m - 3d - d} \times \frac{3s}{l} + d - D\right) \times \frac{1}{12} pl$ = ſolidity of the middle fruſtum of an hyperbolic ſpindle.

But if the generating hyperbola be equilateral, the fruſtum will be $= \left(\frac{3}{2} d^2 - l^2 + \frac{3s}{l} \times \frac{l^2 + D^2 - d^2}{D - d}\right) \times \frac{1}{6} pl$.

Here $(16^2+12^2+28.1708^2)\times 20\times .52359=(256+144+793.5939)\times 20\times .5239=1193.5939\times 20\times .5239=23871.878\times .5239=12499.07660202=$ *ſolidity required.*

2. What is the ſolidity of the fruſtum of any hyperbolic conoid, whoſe greater diameter is 96, leſſer diameter 54, the middle diameter 76.4264392, and the altitude 12.5? *Anſ.* 116160.66.

3. Required the ſolidity of the fruſtum of an hyperbolic conoid, the height being 12, the greateſt diameter 10, the leaſt diameter 6, and the middle diameter $8\frac{1}{2}$? *Anſ.* 667.59.

4. What is the content of the middle fruſtum of an hyperbolic ſpindle, the length being 20, the middle or greateſt diameter 16, the diameter at each end 12, and the diameter at $\frac{1}{4}$ of the length $14\frac{1}{2}$? *Anſ.* 3248.938.

5. Required the content of the ſegment of any ſpindle, its length being 10, the greateſt diameter 8, and the middle diameter 6. *Anſ.* 272.272.

Note. The content of any ſpindle formed by the revolution of a conic ſection about its axis may be found by the following rule:

Add together the ſquares of the greateſt and leaſt diameters, and the ſquare of double the diameter in the middle between the two, and this ſum multiplied by the length and the product again by .1309 will give the ſolidity.

And the rule will never deviate much from the truth when the figure revolves about any other line which is not the axis.

OF

OF THE

REGULAR BODIES.

A Regular body is a solid contained under a certain number of similar and equal plane figures.

The whole number of regular bodies which can possibly be formed is five.

1. The *Tetraedron*, or regular pyramid, which has four triangular faces.

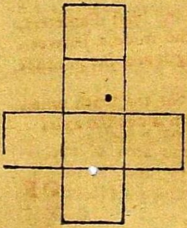
2. The *Hexaedron*, or cube, which has six square faces.

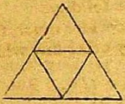
3. The *Octaedron*, which has eight triangular faces.

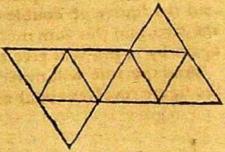
4. The *Dodecaedron*, which has twelve pentagonal faces.

5. The *Icosaedron*, which has twenty triangular faces.

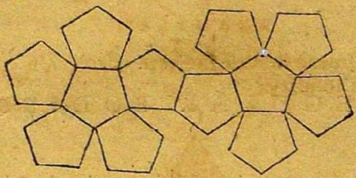
If the following figures are made of pasteboard, and the lines be cut half through, so that the parts may be turned up and glued together, they will represent the five regular bodies here mentioned.

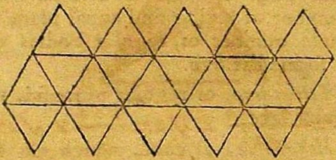






PRO-





PROBLEM I.

To find the solidity of a tetraedron.

RULE.*

Multiply $\frac{1}{12}$ of the cube of the linear side by the square root of 2, and the product will be the solidity.

EXAM-

---

* *Demon.* From one angle C of the tetraedron ABC*n*, let fall the perpendicular C*e*, upon the opposite side, and draw A*e*.

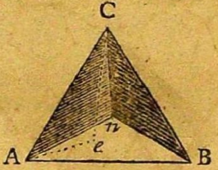
Then $AC^2 - Ae^2 = Ce^2$; and since the point *e* is equally distant from the three angles A, B and *n*, $\frac{1}{3}AC^2 = (\frac{1}{3}AB^2)\ Ae^2$, as is shewn in the demonstration of the rule for regular polygons page 53. Consequently $AC + \frac{1}{3}AC^2 = \frac{2}{3}AC^2 = Ce^2$, or $Ce = AC\sqrt{\frac{2}{3}}$. But the area of the triangle $AnB = \frac{1}{4}AB^2\sqrt{3} = \frac{1}{4}AC^2\sqrt{3}$; and therefore $\frac{1}{3}AC\sqrt{\frac{2}{3}}\ (\frac{1}{3}Ce) \times \frac{1}{4}AC^2\sqrt{3}\ (\triangle AnB) = \frac{1}{12}AC^3\sqrt{2}$.

Q. E. D.

If

EXAMPLES.

1. The linear side of the tetraedron A B C *n* is 4: what is the solidity?



$\frac{4^3}{12} \times \sqrt{2} = \frac{4 \times 4 \times 4}{12} \times \sqrt{2} = \frac{4 \times 4}{3} \times \sqrt{2} = \frac{16}{3} \sqrt{2} = \frac{16}{3} \times 1.414 = \frac{22.624}{3} = 7.5413 =$ *solidity required.*

2. Required the solidity of a tetraedron whose side is 6? *Ans.* 25.452.

PROBLEM II.

To find the solidity of an octaedron.

RULE.*

Multiply $\frac{1}{3}$ of the cube of the linear side by the square root of 2, and the product will be the solidity.

EXAM-

If L be put = length of the linear edge, then will $L^2 \sqrt{3} =$ whole *surface of the tetraedron.*

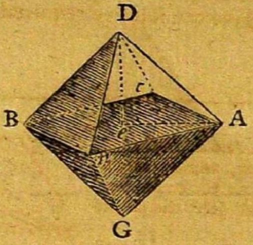
The rule for the hexaedron, or cube, has been given before.

* *Demon.* From the angle D of the octaedron D F G A let fall the perpendicular D *e*.

Then,

EXAMPLES.

1. What is the ſolidity of the octaedron F G A D, whoſe linear ſide is 4?



$\frac{4^3}{3} \times \sqrt{2} = \frac{64}{3} \times \sqrt{2} = 21.333$, &c. $\times \sqrt{2} =$ 21.333, &c. $\times$ 1.452, &c. $=$ 30.16486 $=$ *ſolidity required.*

2. Required the ſolidity of an octaedron whoſe ſide is 8?
Anſ. 243.3568.

PROBLEM III.

To find the ſolidity of a dodecaedron.

Then ſince the ſolid is compoſed of two equal ſquare pyramids, each of whoſe baſes F n A c are equal to the ſquare of the linear ſide A G or A D, we ſhall have F n A $c \times \frac{2}{3}$ D e $=$ A $n^2 \times \frac{2}{3}$ D e $=$ content of the ſolid.

But D e evidently biſects the diagonal F A, and is equal to F e; therefore A $n^2 \times \frac{2}{3}$ D e $=$ A $n^2 \times \frac{2}{3}$ F e $=$ A $n^2 \times \frac{1}{3}$ F A $= \frac{1}{3}$ A $n^2 \times \sqrt{\text{A}n^2 + \text{A}c^2} = \frac{1}{3}$ A $n^2 \sqrt{2\text{A}n^2} = \frac{1}{3}$ A n^3 $\sqrt{2}$. Q. E. D.

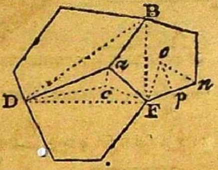
If L $=$ linear ſide as before, then will $2\text{L}^2 \sqrt{3} =$ ſurface of the octaedron.

RULE.

RULE*.

To 21 times the ſquare root of 5 add 47, and divide the ſum by 40: then the ſquare root of the quotient being multiplied by 5 times the cube of the linear ſide will give the ſolidity required.

EXAM-



* *Demon.* Let a be a ſolid angle of the dodecaedron, and ac a perpendicular falling on the equilateral plane BDF. Alſo join the points D, c and c, F.

Then the angle DaF contains 108 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, and the angle aFD contains 36 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$, the radius in both caſes being taken equal to 1.

Therefore, by Trigonometry, $\frac{1}{4}\sqrt{10-2\sqrt{5}} : \frac{1}{4}\sqrt{10+2\sqrt{5}} :: a\text{D} : \text{DF} = a\text{D}\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = a\text{D}\frac{1+\sqrt{5}}{2} = \frac{1}{2}+\frac{1}{2}\sqrt{5}\times a\text{D}.$

Again, ſince c is the centre of DBF, the angles cDF and cFD are each 30°, and the angle DcF $=$ 120°; but the ſine of 30° is $\frac{1}{2}$; and the ſine of 120° is $\frac{1}{2}\sqrt{3}$; whence, by trigonometry, $\frac{1}{2}\sqrt{3} : \text{DF} :: \frac{1}{2} : \text{D}c = \frac{\text{DF}}{\sqrt{3}} = a\text{D}\,\frac{1+\sqrt{5}}{2\sqrt{3}}$; and conſequently $\text{AC} = a\text{D}^2 - \text{DC}^2 = a\text{D}\sqrt{\frac{3-\sqrt{5}}{6}} = a\text{D}\sqrt{\frac{1}{3}-\frac{1}{6}\sqrt{5}}.$

But a perpendicular form a upon the plane BDF muſt paſs through the center of the circumſcribing ſphere, and ac will be the verſed ſine of an arc whoſe chord is aD, and radius equal to that of the ſaid ſphere

Whence $ac : a\text{D} :: a\text{D} : \frac{a\text{D}^2}{a\text{C}} = \frac{a\text{D}^2}{a\text{D}\sqrt{\quad -\frac{1}{6}\sqrt{3}}} = \frac{2a\text{D}}{\sqrt{3}}$

EXAMPLES.

1. The linear ſide of the dodecaedron ABCDE is 3: what is the ſolidity?

$\frac{\sqrt{3}+\sqrt{15}}{2}$ = diameter of the circumſcribing ſphere, and aD $\frac{\sqrt{3}+\sqrt{15}}{4}$ = R = radius of the circumſcribing ſphere.

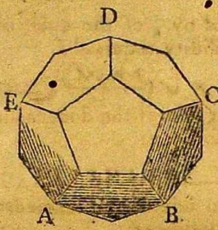
Again, the angle Fon contains 72°, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$; and the angle oFn is 54°, whoſe ſine is $\frac{1+\sqrt{5}}{4}$; whence, by trigonometry, $\frac{1}{4}\sqrt{10+2\sqrt{5}} : \frac{1+\sqrt{5}}{4} :: Fn\,(aD) : oF = aD\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} = aD\sqrt{\frac{5+\sqrt{5}}{10}} = aD\sqrt{\frac{1}{2}+\frac{1}{10}\sqrt{5}}$.

But ſince the radius of the circumſcribing ſphere is the hypothenuſe of a right angled triangle, whoſe two legs are oF and the radius of the inſcribed ſphere, we ſhall have $\sqrt{R^2 - oF^2} = \sqrt{(\frac{1}{4}\sqrt{3+\sqrt{15}})^2 aD^2 - \overline{\frac{1}{2}+\frac{1}{10}\sqrt{5}}\,aD^2)} = aD\sqrt{\frac{25+11\sqrt{5}}{40}} = \sqrt{\frac{5}{8}+\frac{11}{40}\sqrt{5}}\,r$ = radius of the inſcribed ſphere.

And becauſe the ſolid is compoſed of 12 equal pentagonal pyramids, each of whoſe baſes are, by problem the 8th, $= \frac{5aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$; therefore $\frac{60aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}} \times \frac{1}{3}r = \frac{60aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}} \times \frac{aD}{3}\sqrt{\frac{25+11\sqrt{5}}{40}} = 5aD\sqrt{\frac{47+21\sqrt{5}}{40}}$ $\frac{21\sqrt{5}}{40}$ = ſolidity of the dodecaedron. Q. E. D.

If L be put for the linear ſide, then will $15L^2\sqrt{\frac{5+2\sqrt{5}}{5}}$ = ſurface of the dodecaedron.

$\sqrt{21}$



$$\sqrt{\frac{21\sqrt{5}+47}{40}} \times 27 \times 5 = \sqrt{\frac{21 \times 2.23606+47}{40}}$$

$$\times 27 \times 5 = \sqrt{\frac{46.95726+47}{40}} \times 135 = 206.901$$ *solidity required.*

2. The linear side of a dodecaedron is 1; what is the solidity? *Ans.* 7.6631.

PROBLEM IV.

To find the solidity of an icosaedron.

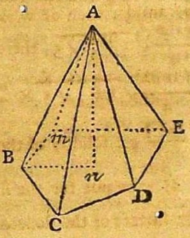
RULE.*

To 3 times the square root of 5 add 7, and divide the sum by 2; then the square root of this quotient being

---

* *Demon.* Let A be a solid angle of the icosaedron, formed by 5 faces, or triangles, whose bases make the pentagon B C D E *m*.

Then, if a perpendicular be demitted from A upon the pentagonal plane B C D E *m*, it will fall into the centre *n*, and B *n*, by the demonstration of the last problem, will be $= AB\sqrt{\frac{5+\sqrt{5}}{10}}$, and the radius of the circle circumscribing one of the faces A B C $= \frac{1}{3} AB\sqrt{3}$.



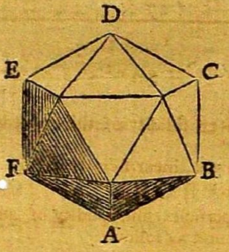
Now

being multiplied by $\frac{5}{6}$ of the cube of the linear side will give the solidity required.

That is $\frac{5}{6}s^3 \times \sqrt{(\frac{7+3\sqrt{5}}{2})}$ = solidity when s = to the linear side.

EXAMPLES.

1. The linear side of the icosaedron ABCDEF is 3: what is the solidity?



$\sqrt{3}$

But the radius of the circumscribing sphere is $R = \frac{BA^2}{2An} = \frac{BA^2}{2\sqrt{AB^2 - Bn^2}} = AB\sqrt{\frac{5+\sqrt{5}}{8}}$, found as in the last problem.

And, since R is the hypothenuse of a right angled triangle, one of whose legs is $\frac{1}{3}AB\sqrt{3}$, the radius of the circle circumscribing the face ABC, and the other r, the radius of the inscribed sphere, we shall have $r = \sqrt{R^2 - \frac{1}{3}(AB\sqrt{3})^2} = \sqrt{\frac{5+\sqrt{5}}{8}AB^2 - \frac{1}{3}AB^2} = AB\sqrt{\frac{7+3\sqrt{5}}{24}}$.

But the solid is composed of 20 equal triangular pyramids, each of whose bases are $= \frac{AB^2}{4}\sqrt{3}$ by problem 8; therefore

$\sqrt{\frac{3\sqrt{5}+7}{2}} \times \frac{5 \times 3^3}{6} = \sqrt{\frac{3 \times 2.23606 + 7}{2}} \times \frac{5 \times 27}{6} = \sqrt{\frac{7.0818+7}{2}} \times \frac{5 \times 9}{2} = \sqrt{\frac{14.0818}{2}} \times \frac{45}{2}$ $= \sqrt{7.0409} \times 22.5 = 2.65 \times 22.5 = 95.625 =$ *solidity required.*

2. Required the solidity of an icosaedron, whose linear side is 1? *Ans.* 2.1816949905.

---

fore $\frac{20AB^2}{4}\sqrt{3} \times \frac{1}{3}r = 5AB^2\sqrt{3} \times \frac{AB}{3}\sqrt{\frac{7+3\sqrt{5}}{24}} = \frac{5}{6}$ $AB^3\sqrt{\frac{7+3\sqrt{5}}{2}} =$ solidity of the icosaedron. Q. E. D.

If L be put for the linear side, then will $5L^2\sqrt{3} =$ surface of the icosaedron.

Note. The superficies, and solidity of any of the 5 regular bodies may be found as follows.

RULE. 1. Multiply the tabular area by the square of the linear edge, and the product will be the superficies.

2. Multiply the tabular solidity by the cube of the linear edge, and the product will be the solidity.

Surfaces and Solidities of the Regular bodies.

N° of Sides.	Names.	Surfaces.	Solidities.
4	Tetraedron	1.73205	0.11785
6	Hexaedron	6.00000	1.00000
8	Octaedron	3.46410	0.47140
12	Dodecaedron	20.64573	7.66312
20	Icosaedron	8.66025	2.18169

OF

OF

CYLINDRIC RINGS.

PROBLEM I.

To find the convex superficies of a cylindric ring.

RULE.*

TO the thickness of the ring add the inner dia meter, and this sum being multiplied by th thickness, and the product again by 9.8696 will giv the superficies required.

EXAMPLES.

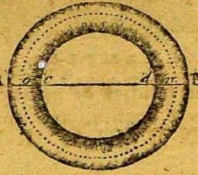
1. The thickness of AC of a cylindric ring is inches, and the inner diameter cd 12 inches: what i the convex superficies?

---

* A solid ring of this kind is only a bent cylinder, an therefore the rules for obtaining its superficies, or solidity, ar the same as those already given. For let AC be any section o the solid perpendicular to its axis on, and then $\overline{AC \times 3.14159}$, &c. = circumference of that section, and $\overline{AC + cd}\,(on) \times 3.14159$, &c. = length of the axis on.

Whence $AC \times 3.14159$, &c. $\times\ \overline{AC + cd} \times 3.14159$, &c. $=$ $\overline{AC + cd} \times AC \times \overline{3.14159, \&c.}^2 = \overline{AC+cd} \times AC \times 9.8696$, &c. $=$ superficies required.

2.



$\overline{12+3} \times 3 \times 9.8696 = 15 \times 3 \times 9.8696 = 45 \times 9.8696 = 444.132 =$ *superficies required.*

2. The thickneſs of a cylindric ring is 4 inches, and the inner diameter 18: what is the convex ſuperficies? *Anſ.* 868.52 *ſquare inches.*

3. The thickneſs of a cylindric ring is 2 inches, and the inner diameter 18: what is the convex ſuperficies? *Anſ.* 394.785 *ſquare inches.*

PROBLEM II.

To find the ſolidity of a cylindric ring.

RULE*.

To the thickneſs of the ring add the inner diameter, and this ſum being multiplied by the ſquare of half the thickneſs, and the product again by 9.8696 will give the ſolidity.

---

* *Demon.* $AC^2 \times .78539$, &c. $= AC^2 \times \frac{3.14159}{4}$, &c. $= \frac{1}{4}AC^2 \times 3.14159$, &c. $=$ area of the ſection AC; and $\overline{AC+CD}$ $(on) \times 3.14159$, &c. $=$ length of the axis on.

Therefore $\overline{AC+CD} \times \frac{1}{4}AC^2 \times \overline{3.14159, \&c.}|^2 = \overline{AC+CD} \times \frac{1}{4}AC^2 \times 9.8698$. Q. E. D.

EXAM-

EXAMPLES.

1. What is the ſolidity of an anchor ring, whoſe inner diameter is 8 inches, and thickneſs in metal 3 inches?

$$\overline{8+3\times\tfrac{3}{2}}|^{2}\times 9.8696 = 11\times 1.5^{2}\times 9.8696 = 11\times 2.25\times 9.8696 = 24.75\times 9.8696 = 244.2726 =$$ *ſolidity required.*

2. The inner diameter of a cylindric ring is 18 inches, and its thickneſs 4 inches: what is the ſolidity?
Anſ. 868.5248.

3. Required the ſolidity of a cylindric ring, whoſe thickneſs is 2 inches, and its inner diameter 12.
Anſ. 138.1744.

4. What is the ſolidity of a cylindric ring, whoſe thickneſs is 4 inches, and inner diameter 26?
Anſ. 789.568.

This figure being only a cylinder bent round into a ring, its ſurface and ſolidity may alſo be found as in the cylinder, namely, by multiplying the axis or length of the cylinder by the circumference of the ring, or ſection, for the ſurface, and by the area of a ſection for the ſolidity.

Thus if $c =$ circumference of the ring, or ſection, $a =$ area of that ſection, and $l =$ length of the axis: then will $cl =$ ſurface of the ring, and $al =$ to its ſolidity. Which rules are the ſame as for the cylinder, and may be eaſily converted into thoſe given in the text.

Theſe rules are indeed ſo obvious, as to render any demonſtration of them altogether unneceſſary.

OF

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimensions of any ball or shell being had from experiment, the weight and dimensions of any other ball or shell, of the same kind, may be found by the following rules.

PROBLEM I.

Having the diameter of an iron shot given, to find its weight.

RULE.*

Take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that eighth, and the sum will be the weight required in pounds, *exactly*.

EXAM-

---

* *Demon.* The weight of an iron shot whose diameter is 4 inches is 9lbs; and as globes are as the cubes of their diameters, therefore $4^3 : 9\text{lbs.} :: D^3$ (diameter being $= D$) : $\frac{9}{64} D^3 =$ weight. But $\frac{9}{64} = \frac{8}{64} + \frac{1}{64} = \frac{1}{8} + \frac{1}{8}$ of $\frac{1}{8}$. Wherefore the weight required is $\frac{1}{8} D^3 + \frac{1}{8}$ of $\frac{1}{8} D^3$.

Q. E. D.

Rule

EXAMPLES.

1. The diameter of an iron ſhot is 3.5 inches; what is its weight?

$$
\begin{array}{r}
3.5 \\
3.5 \\
\hline
175 \\
105 \\
\hline
12.25 \\
3.5 \\
\hline
6125 \\
3675 \\
\hline
8)42.875 \\
\hline
8)5.359 \\
.669 \\
\hline
6.028 = 6lbs.\ nearly.
\end{array}
$$

2. The diameter of an iron ſhot is 6.7 inches: what is its weight? *Anſ.* 42.294*lbs.*

PROBLEM II.

Having the diameter of a leaden ball given to find its weight.

Rule by Logarithms. To 3 times the log. of the diameter, add—1.1480625, and the ſum is the logarithm of the weight required.

RULE.

RULE.*

Take $\frac{1}{3}$ of the cube of the diameter, and from it subtract $\frac{1}{3}$ of this third, and the remainder is the weight required, *nearly*.

EXAMPLES.

1. What is the weight of a leaden ball, whose diameter is 3.3 inches?

$$\begin{array}{r} 3.3 \\ 3.3 \\ \hline 99 \\ 99 \\ \hline 10.89 \\ 3.3 \\ \hline 3267 \\ 3267 \\ \hline 3)35.937 \\ \hline 3)11.979 \\ 3.993 \\ \hline 7.986 \end{array}$$

7.986 = 8*lbs. weight, nearly*.

---

* The weight of a leaden ball, of $4\frac{1}{4}$ inches diameter, is 17lbs. therefore $4.25^3 : 17lbs. :: D^3 : \frac{17}{76.765625} \times D^3 = \frac{2}{9}D^3$ *nearly*, $= (\frac{3}{9} - \frac{1}{9}) \times D^3 = \frac{1}{3}D^3 - \frac{1}{3}$ of $\frac{1}{3}D^3$. Q. E. D.

Rule by Logarithms. To 3 times the log. of the diameter, add — 1.3452821, and the sum will be the logarithm of the weight in pounds *exactly*.

 2. What

2. What is the weight of a leaden ball whofe diameter is 5.24 inches? *Anf. 32 lbs. nearly.*

PROBLEM III.

The weight of an iron ball being given, to find its diameter.

RULE.*

Multiply the weight by 7, and to the product add $\frac{1}{9}$ of the weight, and the cube root of the fum will be the diameter in inches.

EXAMPLES.

1. The weight of an iron ball is 24 lbs. what is the diameter?

$$\begin{array}{r} 24 \\ 7 \\ \hline 168 \\ \tfrac{1}{9} \text{ of } 24 = \quad 2.666 \\ \hline 170.666 \end{array}$$

---

* *Demon.* By problem I. $\frac{9}{64}$ D^3 = weight = w; therefore $D^3 = \frac{64}{9} \times w = \frac{63}{9} w + \frac{1}{9} w = 7w + \frac{1}{9} w$; and confequently $D = \overline{7w + \tfrac{1}{9}w}|^{\frac{1}{3}}$. Q. E. D.

The Rule by Logarithms. To the log. of the weight in pounds, add 0.8519375, and $\frac{2}{3}$ of the fum will be the logarithm of the diameter in inches.

The cube root in any of the following queftions may alfo be extracted by Logarithms.

Let

† *Let the ſuppoſed root be* 5, *whoſe cube is* 125; *then*

125		170.666		
2		2		
250		341.332		
170.666		125		
420.666	:	466.332	::	5
		5		

420.666)2331.660(5.54 *inches* = *cube root of* 170.666, *or the dia. req.*
2103330
―――――
2283300
2103330
―――――
1799700
1682664
―――――
117036

2. The weight of an iron ball is 12lbs. what is the diameter? *Anſ.* 4.403 *inches.*

PROBLEM IV.

Having the weight of a leaden ball given, to find its diameter.

---

† *The rule here uſed for extracting the cube root is as follows:*

Find a root nearly equal to the true one by trial, and then ſay, as twice the cube of the ſuppoſed root added to the given number, is to twice the given number added to the cube of the ſuppoſed root, ſo is the ſuppoſed root to the true root, *nearly*.

For a further account of this method, together with its demonſtration, ſee the *Scholar's Guide to Arithmetic*, page 170.

 RULE.

RULE.*

To 4 times the weight, add half the weight, and $\frac{3}{100}$ of half the weight, and the cube root of this sum will be the diameter, *nearly*.

EXAMPLES.

1. The weight of a leaden ball is 8lbs. what is the diameter?

$$\begin{array}{rl} 8 & \\ 4 & \\ \hline 32 & \\ 4 & = \tfrac{1}{2} \textit{ of } 8 \\ \hline 36 & \\ .12 & = \tfrac{3}{100} \textit{ of } 4 \\ \hline 36.12 & \end{array}$$

Let the supposed root be 3, whose cube is 27; then

---

* *Demon.* By problem II. $D^3 = \frac{76.765625}{17} \times w = 4w + \frac{1}{2}w + \frac{3}{100}$ of $\frac{1}{2}w$, *very nearly*; therefore $D = \overline{4w + \frac{1}{2}w + \frac{3}{100} \text{ of } \frac{1}{2}w}\big|^{\frac{1}{3}}$. Q. E. D.

The Rule by Logarithms. To the log. of the weight in pounds, add 0.6547179, and $\frac{1}{3}$ of the sum will be the logarithm of the diameter, in inches, *exactly*.

```
27              36.12
 2                 2
——             —————
54              72.24
36.12           27.
—————          ——————
90.12    :      99.24   ::   3
                    3
                ——————
       90.12)297.72(3.30=diam. nearly.
             27036
             ——————
              27360
              27036
             ——————
                324
```

2. What is the diameter of a leaden ball whose weight is 12lbs? *Ans.* 3.78 *inches.*

PROBLEM V.

Having the external and internal dimensions of an iron shell given, to find its weight.

RULE.*

Take $\frac{1}{8}$ of the difference of the cubes of the diameters in inches, and $\frac{1}{8}$ of that eighth, and their sum will be the weight in pounds.

---

* *Demon.* The weight is $\frac{9}{64} \times \overline{D^3 - a^3} = (\frac{8}{64} + \frac{1}{64}) \times \overline{D^3 - d^3} = \frac{1}{8} \times \overline{D^3 - d^3} + \frac{1}{8}$ of $\frac{1}{8} \times \overline{D^3 - d^3}$. Q. E. D.

EXAMPLES.

1. What is the weight of a 13 inch iron bomb-shell, the metal being 2 inches thick on a mean?

```
           13
           13
           --
           39
          13
          ---
          169
           13
          ---
          507
         169
         ----
         2197=cube of the external diam.
9×9×9=729=cube of the internal diam.
         ----
       8)1468
         ----
       8)183.5
           22.9
         -----
         206.4=lbs. weight required.
```

2. What is the weight of a 9 inch iron bomb-shell, the metal being $1\frac{1}{2}$ inches thick? *Ans.* 72.14*lbs.*

PROBLEM VI.

To find the number of pounds of powder that a hollow shell will hold.

RULE.

RULE.*

Cube the internal diameter in inches, and divide by 59.32, and the quotient is the weight, *nearly*.

EXAMPLES.

1. How many pounds of powder will a hollow ſhell hold, whoſe internal diameter is 9 inches?

```
         9
         9
        ---
        81
         9
      ------
59.32)729.00(12.28 = pounds nearly.
      5932
      -----
      13580
      11864
      -----
      17160
      11864
      -----
       52960
       47456
       -----
        5504
```

2. How

* *Demon.* The content of the ſhell is $= D^3 \times .5236$; therefore 31.06 (*in. in* 1*lb.*) : 1lb. : : $D^3 \times .5236 : \frac{.5236}{31.06} \times D^3 = \frac{D^3}{59.32}$. Q. E. D.

Or the rule may be expreſſed thus. Cube the diameter, and take $\frac{1}{60}$ part of the reſult; alſo $\frac{1}{60}$ part of this $\frac{1}{60}$; and from the

2. How many pounds of powder will a hollow bomb-ſhell hold, whoſe internal diameter is 13 inches?

Anſ. 37*lbs*, *nearly*.

PROBLEM VII.

To find the dimenſions of a cubical box, to hold a given quantity of powder.

RULE.*

Multiply the weight in pounds by 31.06, and the cube root of the product will give the length of the ſide in inches.

EXAMPLES.

1. What muſt be the length of the ſide of a cubical box that is to hold 15lbs. of powder?

```
31.06
   16
------
15530
3106
------
465.90
```

---

the ſum of theſe two quotients, ſubtract $\frac{1}{3}$ of the laſt quotient, and the remainder is the anſwer, *nearly*.

Rule by Logarithms. To 3 times the log of the diameter in inches, add —2.2267989, and the ſum is the logarithm of the quantity of pounds required.

* *Demon*. As 31.06 inches : 1lb. :: s^3 (the cube of the ſide) : w (the weight); whence $s = \overline{31.06 \times w}|^{\frac{1}{3}}$. Q. E. D.

Rule by Logarithms. To $\frac{1}{3}$ of the log. of the weight in pounds, add 0.4974005, and the ſum will be the logarithm of the ſide in inches.

Let

* *Let* 7.5 *be the ſuppoſed root, whoſe cube is* 421.875; *then*

$$\begin{array}{rcrcl} 421.875 & & 465.9 & & \\ 2 & & 2 & & \\ \hline 843.750 & & 931.8 & & \\ 465.90 & & 421.875 & & \\ \hline 1309.650 & : & 1353.675 & :: & 7.5 \\ & & 7.5 & & \\ \cline{3-3} & & 6768375 & & \\ & & 9475725 & & \end{array}$$

$$1309.650)10152.5625(7.75 = \textit{ſide req.}$$

$$\begin{array}{r} 9167550 \\ \hline 9850125 \\ 9167550 \\ \hline 6825750 \\ 6548250 \\ \hline 277500 \end{array}$$

2. What is the ſide of a cubical box, that is to hold 12lbs. of powder? *Anſ.* 7.19 *inches.*

PROBLEM VIII.

Given the length, breadth, and depth of a rectangular box, to find how many pounds of powder will fill it.

---

* The cube root may here alſo be found by Logarithms.

K 6 RULE.

RULE.*

Multiply the length, breadth, and depth in inches together; and this product being multiplied again by .0322 will give the anſwer in pounds, *nearly*.

But if greater exactneſs be required, ſubtract $\frac{1}{4000}$ part of the product of the three dimenſions from the laſt reſult, and the remainder will be the anſwer, *very nearly*.

EXAMPLES.

1. The length of a rectangular box is 15 inches, the breadth 13 inches, and the depth 5 inches: how many pounds of powder will it hold?

```
     15
     13
   ----
     45
    15
   ----
    195
      5
   ----
    975
  .0322
 ------
   1950
  1950
 2925
 ------
31.3950 = pounds required.
```

2. The

---

* *Demon.* Let L, B, D, = length, breadth, and depth reſpectively.

Then $\frac{LBD}{31.06}$ = weight in pounds $= LBD \times .03219574 = (.0322 - \frac{1}{4000}) \times LBD$. Q. E. D.

Rule

2. The length of a rectangular box is 1 foot, the breadth 9 inches, and the depth 4 inches: how many pounds of powder will it hold? *Ans.* 13.9104*lbs.*

PROBLEM IX.

Given the diameter and length of a hollow cylinder, to find how many pounds of gunpowder will fill it.

RULE.*

Multiply the square of the diameter by $\frac{1}{10}$ of the length; and then take $\frac{1}{4}$ of the product, $\frac{1}{90}$ part of this fourth, and $\frac{1}{90}$ part of this last, and the sum of all these parts will be the answer in pounds, *nearly*.

Rule by Logarithms. Add the logarithms of the length, breadth, and depth in inches, and the constant logarithm —2.5077983 together; and the sum will be the logarithm of the answer in pounds, *exactly*.

* *Demon.* $\frac{D^2 \times .78539 \times L}{31.06} = w = \frac{9}{356} D^2 L$, *nearly*.

But $\frac{9}{356} = \frac{9}{360-4} = \frac{9}{360} + \frac{9}{360} \times \frac{4}{360} + \frac{9}{360} \times \frac{4}{360} \times \frac{4}{360}$, &c. $= \frac{1}{10} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{90} + \frac{1}{4} \times \frac{1}{90} \times \frac{1}{90}$, &c.

Hence $w = D^2 \times \frac{L}{10} \times (\frac{1}{4} + \frac{1}{4} \times \frac{1}{90} + \frac{1}{4} \times \frac{1}{90} \times \frac{1}{90})$, *nearly*. Q. E. D.

The two first terms are near enough for common use.

Rule by Logarithms. To twice the log. of the diameter, add the logarithm of the length of the cylinder, and the constant logarithm—2.4028885, and the sum will be the logarithm of the weight in pounds, *exactly*.

EXAM-

EXAMPLES.

1. The diameter of a hollow cylinder is 4 inches, and the length 1 foot: how many pounds of powder will it hold?

```
      4
      4
     ---
     16 = square of the diameter.
    1.2 = 1/10 of the length.
    ----
     32
    16
    ----
 4)19.2
    ----
    4.8
     .0533 = 1/90 of 4.8
     .0005 = 1/90 of .0533
    ------
    4.8538 pounds the answer.
```

2. The diameter of a hollow cylinder is 6 inches, and its length 10 inches: how many pounds of powder will it hold? *Ans.* 9.1 *lbs.*

PROBLEM X.

Given the diameter of a hollow cylinder, to find what length of it will be filled by a given quantity of powder.

RULE.*

1. Divide the weight in pounds by the square of the diameter in inches. 2. Mul-

---

* *Demon.* By the last problem $l = \frac{356}{9} \times \frac{w}{D^2} = 39\frac{5}{9} \times \frac{w}{D^2} = (40 - \frac{4}{9}) \times \frac{w}{D^2} = (40 - \frac{3}{9} - \frac{1}{9}) \times \frac{w}{D^2} = \frac{40w}{D^2} - (\frac{1}{3} \times \frac{w}{D^2} - \frac{1}{2} \text{ of } \frac{1}{3} \times \frac{w}{D^2})$ *nearly.* Q. E. D.

Rule

2. Multiply the quotient by 40, and from the product subtract $\frac{1}{3}$ of the quotient.

3. From the remainder take $\frac{1}{3}$ of that third, and this last remainder will be the answer, *nearly*.

EXAMPLES.

1. The diameter of a hollow cylinder is 1 foot: what length of it will be filled by 10 pounds of powder?

```
12)10
-----
12).8333
-----
  .0694416=10lbs. divided by the square of 12.
        40
  ---------
 2.7776640
  .0231472=1/3 of .0694416
  ---------
 2.7545168
  .0077157=1/3 of .0231472
  ---------
 2.7468011=height in inches required.
```

2. The diameter of a hollow cylinder is 6 inches: what length of it will be filled by 12lbs. of powder?

Ans. 13.185.

Rule by Logarithms. To the constant logarithm 1.597111, add the logarithm of the given quantity of powder in pounds, and also the arithmetical complement of twice the logarithm of the diameter in inches; and the sum will be the logarithm of the length in pounds.

These rules, and their investigations, were given me by Mr. *Reuben Burrow*.

OF

OF THE

PILING

OF

BALLS AND SHELLS.

IRON *ſhot and ſhells* are uſually piled in horizontal courſes, in a pyramidical or wedge like form, the baſe being either an equilateral triangle, a ſquare or a rectangle. In the triangle and ſquare, the pile finiſhes in a ſingle ball; but in the rectangle, the top is a ſingle row of balls.

In triangular and ſquare piles the number of horizontal rows or courſes is always equal to the number of ſhot on one ſide, in the bottom row. But, in rectangular piles, the number of courſes is equal to the number of ſhot in the breadth of the bottom row; and the number in the top row, leſs one, is the difference between the number in the length and breadth of the bottom row.

PROBLEM I.

To find the number of ſhot in a finiſhed triangular pile.

RULE.*

Multiply the number in one ſide of the bottom row plus two, by that number plus one; and this product being

* *Demon.* In triangular piles, each horizontal courſe is a triangular number, produced by taking the ſucceſſive ſums of the

being again multiplied by $\frac{1}{6}$ of the number in the ſaid row, will give the anſwer required.

EXAMPLES.

1. Required the number of ſhot in a finiſhed triangular pile, the number in one ſide of the baſe being 20?

22 = *number plus* 2
21 = *number plus* 1
—
22
44
—
462
20
—
6)9240
—
1540=*number of ſhot in the whole pile.*

2. Required the number of ſhot in a finiſhed triangular pile, the number in one ſide of the baſe being 40? *Anſ.* 11480.

---

the numbers 1; 1, 2; 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5, &c. and the whole number of ſhot in ſuch a pile is equal to the ſum of all thoſe triangular numbers, taken as far, or to as many terms, as are equal to the number in one ſide of the bottom courſe.

Thus, if 1, 2, 3, 4, 5, 6, 7, &c. be the natural numbers, then will 1, 3, 6, 10, 15, 21, 28, be the triangular numbers; or the number of ſhot in each courſe from the top.

But the ſum of the ſeries 1+3+6+10+15+21+28, &c. to n terms is $=\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$ (by *Simpſon*'s *Algebra*, page 216) $=\overline{n+2}\times\overline{n+1}\times\frac{n}{6}$; which is the ſame as the rule. Q. E. D.

PRO-

PROBLEM II.

To find the number of ſhot in a finiſhed ſquare pile.

RULE.*

Multiply the number in one ſide of the bottom row plus one, by double that number plus one; and this product being again multiplied by $\frac{1}{6}$ of the ſaid number, will give the anſwer required.

EXAMPLES.

1. Required the number of ſhot in a finiſhed ſquare pile, the number in one ſide of its baſe being 20.

41 = *double of* 20, *plus* 1
21 = 20 *plus* 1

41
82

861
20

6)17220

2870 = *number of ſhot in the pile.*

2. Required

---

* *Demon.* In ſquare piles, each horizontal courſe is a ſquare number, produced by taking the ſquare of the number in its ſide; and the whole number of ſhot in ſuch a pile is equal to the ſum of all thoſe ſquares, beginning at one, and proceeding as far as the number in the ſide of the bottom courſe.

Thus, if 1, 2, 3, 4, 5, 6, 7, &c. be the natural numbers, then will 1, 4, 9, 16, 25, 36, 49, &c. be the ſquare numbers, or the number of ſhot in each courſe from the top.

But

2. Required the number of ſhot in a finiſhed ſquare pile, one ſide of the lower tier having 40 ſhot in it.

Anſ. 22140.

PROBLEM III.

To find the number of ſhot in a finiſhed rectangular pile.

RULE.*

1. To twice the length of the uppermoſt row add the number of courſes leſs one, and multiply the ſum by the number of courſes plus 1.

But the ſum of the ſeries $1+4+9+16+25+36+49$, &c. to n terms is $= \frac{n \times (n+1) \times (2n+2)}{6}$ (by *Simpſon*'s *Algebra*, page 206) $= \overline{n+1} \times \overline{2n+2} \times \frac{n}{6}$, which is the ſame as the rule. Q. E. D.

* *The inveſtigation of this rule* is given by *Mr. Simpſon*, in page 209 of his Algebra thus: Let $m+1$ and $p+1$ repreſent the length and breadth of the uppermoſt rank, and n the number of ranks one above another.

Then will $\overline{m+1} \times \overline{p+1} + \overline{m+2} \times \overline{p+2} + \overline{m+3} \times \overline{p+3}$, &c. be the number of ſhot in each of the rectangular courſes of the pile.

And $\overline{m+1} \times \overline{p+1} + \overline{m+2} \times \overline{p+2} + \overline{m+3} \times \overline{p+3} \ldots \ldots + \overline{m+n} \times \overline{p+n} =$ number of ſhot in the pile, whether it be whole, or broken.

But $\overline{m+1} \times \overline{p+1} + \overline{m+2} \times \overline{p+2} + \overline{m+3} \times \overline{p+3} \ldots\ldots + \overline{m+n} \times \overline{p+n} = n \times (mp + \frac{n+1}{2} \times \overline{m+p} + \frac{(n+1) \times (2n+1)}{6}$ $= \frac{n}{4} \times (\overline{2m+n+1} \times \overline{2p+n+1} + \frac{(n+1) \times (n-1)}{3}$ which is the ſame as the rule. Q. E. I.

2. Mul-

2. Multiply the number of courses less one by that number more one, and add $\frac{1}{3}$ of this product to the former.

3. Take $\frac{1}{4}$ of this last sum and multiply it by the number of courses, and it will give the answer required.

EXAMPLES.

1. How many shot are there in a finished rectangular pile of 15 courses, the number in the uppermost row being 32?

```
   32
    2
   --
   64 = twice the uppermost row.
   14 = number of courses less 1.
   --
   78
   16
  ---
  468
  78
 ----
 1248
 ----
   14 = number of courses less 1.
   16 = number of courses more 1.
   --
   84
  14
  ---
3)224
-----
  74⅔
 1248
-----
 1322⅔                    4)1322⅔
```

$$
\begin{array}{r}
4)1322\frac{2}{3} \\
\hline
330\frac{2}{3} \\
15 \\
\hline
1660 \\
330 \\
\hline
4960
\end{array}
$$
4960 = *number of shot in the pile.*

2. How many ſhot are there in a finiſhed rectangular pile of 20 courſes, the number in the top row being 40? *Anſ.* 11060.

The number of ſhot in any broken pile may be obtained, by computing what the whole finiſhed pile would contain, and alſo what the pile taken away contained; in which caſe the remainder muſt be the number in the unfiniſhed part as required.

The rules for finding the number of ſhot in the different kinds of piles, may be expreſſed algebraically thus:

$\frac{n}{6} \times \overline{n+1} \times \overline{n+2} =$ N° in the triangular pile.

$\frac{n}{6} \times \overline{n+1} \times \overline{2n+1} =$ N° in the ſquare pile.

$\frac{n}{6} \times (n+1) \times (3m-n+1) =$ N° in the rectangular pile.

Where n denotes the number in one ſide of the bottom row, for the triangular and ſquare piles. And in the rectangular pile it denotes the breadth of the bottom row, and m its length. Which rule, for the latter caſe, is ſomething more convenient than that given in the text.

OF

ARTIFICERS WORK.

ARTIFICERS eſtimate, or compute, the value of their works by different meaſures, *viz* *.

1. *Glazing*, and *Maſon*'s flat work, &c. by the foot.

2. *Painting*, *Plaſtering*, *Paving*, &c. by the yard.

3. *Flooring*, *Partitioning*, *Roofing*, *Tiling*, &c. by the ſquare of 100 feet.

4. *Brickwork*, &c. by the rod of $16\frac{1}{2}$ feet, whoſe ſquare is $272\frac{1}{4}$.

The meaſures made uſe of in theſe works are contained in the following table:

12 inches	make	1 lineal foot
144 ſquare inches	make	1 ſquare foot
9 ſquare feet	make	1 ſquare yard
100 ſquare feet	make	a ſquare
$272\frac{1}{4}$ ſquare feet, or $30\frac{1}{4}$ ſquare yards	make	1 rod, perch, or ſquare pole.

* The beſt method of taking the dimenſions of all ſorts of artificers work is by feet, tenths and hundredths; becauſe the computations may then be performed by common multiplication, or by the ſliding rule, hereafter deſcribed.

OF

BRICKLAYERS WORK*.

BRICKLAYERS compute or value their work at the rate of a brick and a half thick, and, if a wall be more or lefs than this ftandard, it muft be reduced to it, as follows:

---

* *Note.* In practice it is ufual to divide the fquare feet by 272 only, omitting the $\frac{1}{4}$.

The ufual way to take the dimenfions of a building, is to meafure half round its middle, on the outfide, and half round it on the infide; and this will give the true compafs, in which the thicknefs of the wall is included.

When the height of the building is unequal, take feveral different altitudes, and their fum, being divided by the number you have taken, may be confidered as the mean height.

To meafure a chimney ftanding by itfelf, without any party-wall adjoining; girt it about for the length, and reckon the height of the ftory for the breadth; but if it ftands againft a wall, you muft meafure it round to the wall for the girt, and take the height as before.

When the chimney is wrought upright from the mantle-tree to the cieling, the thicknefs muft always be the fame with the jaumbs; and nothing is ever deducted for the vacancy between the floor and the mantle-tree, becaufe of the gathering of the breaft and wings to make room for the hearth in the next ftory.

To meafure chimney fhafts, or that part which appears above the roof; girt them with a line, about the leaft place for the length, and take the height for the breadth; and if they be four inches thick, fet down the thicknefs at one brick-work; but if they be nine inches thick, reckon it at a brick and a half, in confideration of the plaftering and fcaffolding.

RULE.

Multiply the ſuperficial content of the wall in feet, by the number of half bricks in the thickneſs, and $\frac{1}{3}$ of that product will be the content required.

EXAMPLES.

1. How many ſquare rods are there in a wall 52½ feet long, 12 feet 9 inches high, and 2½ bricks thick?

By Decimals.

52.5 *length*
12.75 = *height*
———
2625
3675
1050
525
———
272)669.375(2.4609
544 — 5 *half bricks*
———
1253 — 3)12.3045
1088 — *ro. fe. in.*
——— 4.1015 = 4 27 7 *the anſwer.*
1657
1632
———
2550
2448
———
102

By

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the value of

By Cross Multiplication.

	feet	*in.*		
	52	6		
	12	9		
	630	0		
	39	4	6	
272)	669	4	6	
	2	124	4	
			5	
3)	12	82	8	
	4	27	7	*as before.*

2. How many square rods are there in a wall $62\frac{1}{2}$ feet long, 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick?

ro. fe. in. p.
Ans. 5 167 9 4

3. If each side wall of a building be 45 feet long on the outside, each end wall 15 feet broad on the inside, the height of the building 20 feet, and the gable at each end of the wall 6 feet high, the whole being 2 bricks thick; what is the true content in standard rods? *Ans.* 12.1761.

---

of their workmanship is added to the bill at the stated rate agreed on.

There are also other allowances to be made to the workman, such as those for returns or angles made by two adjoining walls, and double measure for feathered gable ends, &c.

All ornamental work is generally valued by the foot square; such as arches, doors, architraves, frizes, cornices, &c. But carved mouldings, &c. are often agreed for by the running foot, or lineal measure.

OF
MASONS WORK.

TO Mafonry belongs all forts of ftone work, and the meafure made ufe of is a foot, either fuperficial or folid.

Walls, blocks of marble or ftone, columns, &c. are meafured by the folid foot; and pavements, flabs, chimney-pieces, &c. by the fuperficial foot*.

EXAMPLES.

1. Required the folid content of a wall whofe length is 48 feet 6 inches, its height 10 feet 9 inches, and thicknefs 2 feet.

By Decimals.

```
    48.5
    10.75
   ------
    2425
   3395
  4850
  -------
  521.375
        2
  -------
 1042.750 the anfwer.
```

By

* Solid meafure is principally ufed for materials, and the fuperficial for workmanfhip.—In the folid meafure the true length, breadth, and thicknefs, are taken, and multiplied continually together. And in the fuperficial meafure, the length and breadth of every part of the projection muft be taken, as it appears without the general upright face of the building.

All

By Cross Multiplication.

feet	in.	
48	6	
10	9	
485	0	
36	4	6
521	4	6
2	0	0
1042	9	0 *the same as before.*

2. Required the solid content of a wall whose length is 53 feet 6 inches, its height 12 feet 3 inches, and its thickness 2 feet. *Ans.* 1310 *feet* 9 *in.*

3. What is a marble slab worth, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches, at 6s. per foot? *Ans.* 3*l.* 1*s.* 5*d.*

4. What is the solid content of a wall, whose length is 60 feet 9 inches, its height 10 feet 3 inches, and its thickness $2\frac{1}{2}$ feet? *Ans.* 1556.71875 *feet.*

5. In a chimney-piece, suppose the

	fe.	in.
Length of the mantle and slab each, -	4	6
Breadth of both together, - -	3	2

	fe.	in.
Length of each jamb, - -	4	4
Breadth of both together, -	1	9

What will be the content of the chimney-piece? *Ans.* 21 *feet* 10 *in.*

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed; but this deduction is only to be made with regard to materials, for the value of the workmanship is to be added to the workman's bill at the stated rate only.

OF

CARPENTERS AND JOINERS WORK*.

CARPENTERS and Joiners work is that of flooring, partitioning, roofing, &c. and is measured by the square of 100 feet.

EXAM-

---

* *Note.* Large and plain articles are usually measured by the foot, or yard, &c. square; but inriched mouldings, and some other articles, are often estimated by running or lineal measure, and some things are rated by the piece.

In measuring of joists it is to be observed, that only one of their dimensions is the same with that of the floor, and the other will exceed the length of the room by the thickness of the wall, and $\frac{1}{3}$ of the same, because each end is let into the wall about $\frac{2}{3}$ of its thickness.

No deductions are made for hearths, on account of the additional trouble and waste of materials.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In measuring of Joiners work, the string is made to ply close to every part of the work over which it passes.

The measuring of centering for cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length; but in groin-centering, it is usual to allow double measure, on account of their extraordinary trouble.

In roofing, the length of the house in the inside, together with $\frac{2}{3}$ of the thickness of one gable, is to be considered as the length, and the breadth is equal to double the length of a string which is stretched from the ridge down to the rafter, along the eaves-board, till it meets with the top of the wall.

For

EXAMPLES.

1. If a floor be 57 feet 3 inches long, and 28 feet 6 inches broad: how many squares will it contain?

By Decimals.

```
     57.25
      28.5
    ------
     28625
    45800
   11450
   -------
   16|31.625
```

By

For stair-cases, take the breadth of all the steps, and make a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends, and by the breadth is to be understood the girt of its two upper surfaces, or the tread and riser.

For the ballustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel post, for the length: and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the breadth.

For wainscotting, take the compass of the room for the length; and the height from the floor to the cieling, making the string ply close into all the mouldings, for the breadth.—Out of this must be made deductions for windows, doors, and chimneys, &c. but workmanship is counted for the whole, on account of the extraordinary trouble.

For doors, it is usual to allow for their thickness, by adding it into both the dimensions of length and breadth, and then multiplying them together for the area.—If the door be pannelled on both sides, take double its measure for the workmanship;

By Cross Multiplication.

feet	*in.*	
57	3	
28	6	
1603	0	
28	7	6
16|31	7	6

2. A floor is 53 feet 6 inches long, and 47 feet 9 inches broad: how many squares will it contain?

Ans. 25 *sq. and* 54 *feet.*

3. A partition is 91 feet 9 inches long, and 11 feet 3 inches broad: how many squares will it contain?

Ans. 10 *sq. and* 32 *feet.*

4. If a house within the walls be 44 feet 6 inches long, and 18 feet 3 inches broad: how many squares of roofing will cover it? *Ans.* 12 *sq. and* 18 *feet.*

5. If a house measure within the walls 52 feet 8 inches in length, and 30 feet 6 inches in breadth, and the roof be of a true pitch, what will it cost roofing at 10*s*. 6*d*. per square? *Ans.* 12*l*. 12*s*. 11¾*d*.

---

manship; but if one side only be pannelled, take the area and its half for the workmanship.

For the surrounding architrave, gird it about the outermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.

Window-shutters, bases, &c. are measured in the same manner.

In the measuring of roofing for workmanship alone, all holes for chimney shafts and sky-lights are generally deducted.

But in measuring for work and materials, they commonly measure in all sky-lights, luthern-lights, and holes for the chimney shafts, on account of their trouble and waste of materials.

OF

SLATERS AND TILERS WORK.

IN these works, the content of a roof is found by multiplying the length of the ridge by the girt from eve to eve; and, in slating, allowance must be made for the double row at the bottom.

In taking the girt, the line is made to ply over the lowest row of slates, and returned up the under-side till it meet with the wall or eaves board; but in tiling, the line is stretched down only to the lowest part, without returning it up again.

Double measure is generally allowed for hips, vallies, gutters, &c. but no deductions are made for chimnies.

EXAMPLES.

1. The length of a slated roof is 45 feet 9 inches, and its girt 34 feet 3 inches: what is the content?

By Decimals.

```
      45.75
      34.25
     ------
     22875
     9150
    18300
   13725
   --------
9)1566.9375
   --------
    174.104
```

By Cross Multiplication.

	feet	*in.*		
	45	9		
	34	3		
	1555	6		
	11	5	3	
9)	1566	11	3	
	174	0	11	3

Ans. 174 *yards.*

2. What will the tiling a barn cost at 25*s.* 6*d.* per square, the length being 43 feet 10 inches, and the breadth 27 feet 5 inches, on the flat, the eave-boards projecting 16 inches on each side? *Ans.* 24*l.* 9*s.* $5\frac{1}{2}$*d.*

In angles formed in a roof, running from the ridge to the eaves, that angle of the roof, which bends inwards, is called a *valley*; and the angle bending outwards is called a *hip*. And in tiling and slating, it is common to add the length of the vallies to the content in feet; and sometimes also the hips are added.

In slating it is common to reckon the breadth of the roof 2 or 3 inches broader than what it measures, because the first row is almost covered by the second; and this is done sometimes when a roof is tiled.

Note. Sky-lights and chimney-shafts are always deducted; but they seldom deduct luthern lights, or garret windows on the roof; for the covering them is reckoned equal to the hole in the roof.

In all works of this kind the content is computed, either in yards of 9 square feet, or in squares of a hundred feet, and the same allowance of hips and vallies is to be made as in roofing.

It is customary to reckon the flat and half of any building within the walls, for the measure of the roof of that building, when the roof is of a *true pitch*, or so that the rafters are $\frac{3}{4}$ of the breadth of the building.

PLASTERERS

PLASTERERS WORK.

PLASTERERS work is of two kinds, viz. plaſtering upon laths, called cieling; and plaſtering upon walls, called rendering; and theſe different kinds muſt be meaſured ſeparately, and their contents collected into one ſum.

Note. Proper deductions muſt be made for doors, windows, &c.—And in meaſuring between quarters, there is commonly $\frac{1}{5}$ part of the whole area allowed; but when rendering between quarters is whitened or coloured, there is $\frac{1}{5}$ part to be added to the whole for the ſides of the quarters and braces.

EXAMPLES.

If a cieling be 59 feet 9 inches long, and 24 feet 6 inches broad: how many yards does it contain?

By Decimals.

```
     59.75
      24.5
   -------
     29857
    23900
   11950
   -------
9)1463.875
   -------
   162.652
```

Plaſterers plain work is meaſured by the ſquare foot, or yard of 9 ſquare feet, and enriched mouldings, &c. by running or lineal meaſure.

By Cross Multiplication.

feet	*in.*	
59	9	
24	6	
1434	0	
29	10	6
9)1453	10	6
162 5	10	6

Ans. 162 *yards* 5 *feet.*

2. If the partitions between rooms be 141 feet 6 inches about, and 11 feet 3 inches high; how many yards do they contain? *Ans.* 176.87.

3. There is a quantity of partitioning that measures 234 feet 8 inches about, and 14 feet 6 inches high, and is rendered between quarters; the lathing and plastering of which will be 8*d.* per yard, and the whiting 2*d.* per yard: what will the whole come to? *Ans.* 13*l.* 17*s.* $2\frac{3}{4}$*d.*

4. The length of a room is 14 feet 5 inches, its breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, whose girt is $8\frac{1}{2}$ inches, and its projection 5 inches from the wall on the upper part next the cieling: what will be the quantity of plastering, supposing there are no deductions but for one door, whose size is 7 feet by 4 feet? *Ans.* 53 *yards* 5 *feet* 3 *inches, of rendering,* 18 *yards* 5 *feet* 6 *inches, of cieling, and* 39 *feet* $0\frac{1}{11}$ *nches, of cornice.*

The work is chiefly of two kinds; viz. plastering upon laths, called cieling; and plastering upon walls, called rendering; all the different kinds are to be measured separately; and then the contents of the same kind, or the same number of coats must be collected together into one sum for the content of the whole.

OF

OF

PAINTERS WORK.

PAINTERS work is meaſured in the ſame manner as that of Joiners; and in taking the dimenſions, the line muſt be forced into all the mouldings and corners.

Windows are done at ſo much a-piece, and in carved mouldings, &c. it is cuſtomary to allow double the uſual meaſure.

EXAMPLES.

1. If a room be painted, whoſe height is 16 feet 6 inches, and its compaſs 97 feet 9 inches: how many yards does it contain?

By Decimals.

```
       97.75
       16.5
    --------
       48875
      58650
      9775
    --------
   9)1612.875
    --------
     179.208
```

Baluſtrades and moſt other works of that kind are meaſured as in Joiners work, excepting for doors, window-ſhutters, &c. where the Joiner is only allowed the area and half area, but the Painter has always double the area of one ſide, becauſe every part that is painted muſt be meaſured.

By Cross Multiplication.

feet	in.	
97	9	
16	6	
1564	0	
48	10	6
9)1612	10	6
179 1	10	6

Ans. 179 *yards* 1 *foot*.

2. The height of a room is 14 feet 10 inches, and the circumference 21 feet 8 inches, how many square yards does it contain? *Ans.* 35.

3. Suppose a room, that was to be painted at 8*d.* *per* yard, measures as follows: The height is 11 *fe.* 7 *in.* the girt or compass 74 *fe.* 10 *in.* the door 7 *fe.* 6 *in.* by 3 *fe.* 9 *inches*; five window shutters, each 6 *fe.* 8 *in.* by 3 *fe.* 4 *in.* the breaks in the windows 14 *in.* deep, and 8 *fe.* high; the chimney 6 *fe.* 9 *in.* by 5 *fe.* a closet, the height of the room, $3\frac{1}{2}$ *fe.* deep, and $4\frac{3}{4}$ *fe.* in front, with shelving, at 22 *fe.* 6 *in.* by 10 *in.* the shutters, door and shelves, being all coloured on both sides: what will the whole come to? *Ans.* 4*l.* 18*s.* 9*d.*

Note. Painters take their dimensions with a string, and measure from the top of the cornice to the floor, girting the string over all the mouldings and swellings; and their price is generally proportioned to the number of times they lay on their colour.

All work of this kind is done by the square yard, and every part where the colour lies must be measured, and estimated in the general account of the work.

Deductions are to be made for chimnies, casements, &c. and the price is generally proportioned to the number of times they lay on their colour.

OF

OF

GLAZIERS WORK.

GLAZIERS take their dimensions in feet, inches, and parts, and estimate their work by the square foot.

In taking the length and breadth of a window, the cross-bars between the panes are always included.

Windows of every form are measured as if they were squares, and the greatest lengths and breadths are constantly to be taken, on account of the waste attending the cutting.

EXAMPLES.

1. If a pane of glass be 2 feet 8 inches and 3 quarters long, and 1 foot 4 inches 1 quarter broad: how many feet does it contain?

By Decimals.

The decimal of $8\frac{3}{4}$ *inches is* .729

And that of $4\frac{1}{4}$ *inches is* .354

$$
\begin{array}{r}
2.729 \\
1.354 \\
\hline
10916 \\
13645 \\
8187 \\
2729 \\
\hline
3.695066 \\
12 \\
\hline
8.340792
\end{array}
$$

Glaziers generally measure their work to a quarter of an inch, and all circular, triangular, &c. windows, are measured as if they were squares.

By

By Cross Multiplication.

feet	in.	pa.		
2	8	9		
1	4	3		
2	8	9		
	10	11	0	
		8	2	3
3	8	4	2	3

Ans. 3 *feet* 8 *in.* 4 *pa.*

2. What will the glazing a triangular sky-light come to at 10*d.* per foot, supposing the base to be 12 feet 6 inches, and the perpendicular height 16 feet 9 inches? *Ans.* 4*l.* 7*s.* $2\frac{3}{4}$*d.*

3. There is a house with 3 tier of windows, three in a tier; the height of the first tier is 7 feet 10 inches; of the second 6 feet 8 inches; of the third 5 feet 4 inches, and the breadth of each 3 feet 11 inches: what will the glazing come to at 14*d.* per foot? *Ans.* 13*l.* 11*s.* $10\frac{1}{2}$*d.*

4. There is a house with 3 tier of windows, three in a tier; the height of those in the lower tier being 7 *fe.* 9 *in.* of those in the middle 6 *fe.* 6 *in.* and of those in the upper tier 5 *fe.* $3\frac{1}{4}$ *in.* and having also a circular window above the door, the greatest height of which is 1 foot $10\frac{1}{2}$ *in.* and the common breadth of all the windows 3 *fe.* 9 *in.* what will the glazing come to at $13\frac{1}{2}$*d.* per foot? *Ans.* 12*l.* 14*s.* $11\frac{1}{2}$*d.*

Note. Windows are sometimes measured by taking the dimensions of one pane, and multiplying it continually by the number of panes; and no allowance is ever made for round or oval windows, as the trouble of cutting them to those shapes is more than the value of the glass omitted.

OF

PAVIOURS WORK*.

PAVIOURS work is done by the ſquare yard, and the content is found by multiplying the length by the breadth.

Or if the dimenſions be taken in feet, and the area be found in the ſame meaſure, the reſult being divided by 9, will give the number of ſquare yards required.

EXAMPLES.

1. What will the paving a rectangular court-yard come to at 3*s.* 2*d.* per yard, ſuppoſing the length to be 27 feet 10 inches, and the breadth 14 feet 9 inches?

By Croſs Multiplication.

feet	*in.*	
27	10	
14	9	
389	8	
20	10	6
9)410	6	6
45	5	6 6

* Plumbers work is generally done by the pound, or hundred weight, and the price is regulated according to the value of the lead at the time the contract is made, or when the work is performed.

				45	5 6	6 *at* 3*s.* 2*d.*
	s.	*d.*				
	2	0	$\frac{1}{10}$	45		
	1	0	$\frac{1}{2}$	4	10	
		2	$\frac{1}{6}$	2	5	
f.	*in.*	*p.*				
3	0	0	$\frac{1}{3}$		7	6
1	0	0	$\frac{1}{3}$		1	0$\frac{1}{2}$
1	0	0	$\frac{1}{1}$		0	4
0	6	0	$\frac{1}{2}$		0	4
0	0	6	$\frac{1}{12}$		0	2
					0	0
				7	4	4$\frac{1}{2}$ *the answer.*

2. A rectangular court-yard is 42 feet 9 inches long, and 68 feet 6 inches in depth, and a foot-way goes quite through it, of 5 feet 6 inches in breadth; the foot-way is laid with stone at 3*s.* 6*d.* per yard, and the rest with pebbles at 3*s.* per yard: what will the whole come to? *Ans.* 49*l.* 17*s.* 0$\frac{1}{4}$*d.*

Sheet-lead, used in roofing, guttering, &c. is generally between 7 and 12lbs. weight to the square foot.

The following table will shew the weight of a square foot to each of these thicknesses.

Thick.	*lbs. sq. foot.*	*Thick.*	*lbs. sq. foot.*	*Thick.*	*lbs. sq. foot.*
$\frac{1}{8}$	7.373	.15	8.848	.18	10.618
.13	7.668	.16	9.438	.19	11.207
.14	8.258	$\frac{1}{6}$	9.831	$\frac{1}{5}$	11.797
$\frac{1}{7}$	8.427	.17	10.028	.21	12.387

OF

OF

VAULTED AND ARCHED ROOFS.

ARCHED roofs are either *vaults*, *domes*, *ſaloons*, or *groins*.

Vaulted roofs are formed by arches ſpringing from the oppoſite walls, and meeting in a line at the top.

Domes are made by arches ſpringing from a circular or polygonal baſe, and meeting in a point at the top.

Saloons are formed by arches connecting the ſide walls to a flat roof, or cieling, in the middle.

Groins are formed by the interſection of vaults with each other.

Domes and Saloons rarely occur in the practice of meaſuring, but vaults and groins cover the cellars of moſt houſes.

Vaulted roofs are generally one of the three following ſorts:

1. *Circular roofs*, or thoſe whoſe arch is ſome part of the circumference of a circle.

2. *Elliptical roofs*, or thoſe whoſe arch is ſome part of the circumference of an ellipſis.

3. *Gothic roofs*, or thoſe which are formed by two circular arcs that meet in a point directly over the middle of the breadth, or ſpan of the arch.

PROBLEM I.

To find the ſolid content of circular, elliptic, or gothic vaulted roofs.

RULE.

RULE.*

Multiply the area of one end by the length of the roof, and the product will be the solidity required.

EXAMPLES.

1. What is the solid content of a semi-circular vault, whose span is 40 feet, and its length 120 feet?

```
        .7854
         1600=square of 40.
      ---------
      4712400
      7854
   ----------
   2)1256.6400
   ----------
      628.32=area of the end.
         120=length.
   ----------
    75398.40=solidity required.
```

2. Required the solidity of an elliptic vault, whose span is 40 feet, height 12 feet, and length 80?

Ans. 30159.36 *feet.*

3. What is the solid content of a gothic vault, whose span is 48, the chord of its arch 48, the distance of the arch from the middle of the chord 18, and the length of the vault 18? *Ans.* 136228.044.

---

* *To find the solidity of the materials in either of the arches.*

Rule. From the solid content of the whole arch take the solid content of the void space, and the remainder will be the solidity of the arch.

PRO-

PROBLEM II.

To find the concave, or convex surface, of circular, elliptic, or gothic vaulted roofs.

RULE.*

Multiply the length of the arch by the length of the vault, and the product will be the superficies required.

EXAMPLES.

1. What is the concave surface of a semi-circular vault, whose span is 40 feet, and its length 120?

$$
\begin{array}{r}
3.1416 \\
40 \\
\hline
2)125.6640 \\
\hline
62.832 = \textit{length of the arch.} \\
120 \\
\hline
7539.840 = \textit{concave surface required.}
\end{array}
$$

PROBLEM III.

To find the solid content of a dome; its height, and the dimensions of its base being known.

---

* The convex surface of a vault may be found by stretching a string over it; but for the concave surface this method is not applicable, and therefore its length must be found from proper dimensions.

RULE.

RULE.*

Multiply the area of the baſe by $\frac{2}{3}$ of the height, and the product will be the ſolidity.

EXAMPLES.

1. What is the ſolid content of a ſpherical dome, the diameter of whoſe circular baſe is 60 feet?

```
      .7854
       3600=square of 60.
     -------
    4712400
    23562
    -------
  2827.4400=area of the baſe.
         20=2/3 of the height (30).
  ---------
 56548.8000=ſolidity required.
```

2. In an hexagonal ſpherical dome, one ſide of the baſe is 20 feet; what is the ſolidity?

Anſ. 12000 *feet.*

PROBLEM IV.

To find the ſuperficial content of a ſpherical dome.

RULE.†

Multiply the area of the baſe by 2, and the product will be the ſuperficial content required.

---

* Domes and ſaloons are of various figures, but they are things that ſeldom occur in the practice of meaſuring.

† *The practical rule for elliptical domes is as follows:*

Rule. Add the height to half the diameter of the baſe, and this ſum multiplied by 1.5708 will give the ſuperficial content *nearly*.

EXAM-

EXAMPLES.

1. What will the painting an hexagonal ſpherical dome come to at 1*s*. per yard; each ſide of the baſe being 20 feet?

2.598076 = *area of a hexagon whoſe ſide is* 1.
400 = *ſquare of* 20.

1039.230400 = *area of the baſe.*
2

9)2078.460800 = *ſuperficial content required.*

2.0)230.940088

11.5470044 = 11*l*. 10*s*. 11*d*. *the expence of* [*painting.*

PROBLEM V.

To find the ſolid content of a ſaloon.

RULE.*

1. Multiply the height of the arc, its projection, $\frac{1}{4}$ of the perimeter of the cieling, and 3.1416 continually together, and call the product A.

2. From

---

* *To find the ſuperficial content of a ſaloon.*

1. Find the area of the flat part of the cieling.

2. Find the convex ſurface of a cylinder, or cylindroid, whoſe length is equal to $\frac{1}{4}$ the perimeter of the cieling, and its diameters to twice the height and twice the projection of the arch.

3. Find the ſuperficial content of a dome of the ſame figure as the arch, and whoſe baſe is either a ſquare, or a figure ſimilar

2. From a ſide or diameter of the room take a like ſide or diameter of the cieling, and multiply the ſquare of the remainder by the proper factor (page 54) and this product again by $\frac{2}{3}$ of the height, and call the laſt product B.

3. Multiply the area of the flat cieling by the height of the arch, and this product added to the ſum of A and B will give the content required.

EXAMPLES.

1. What is the ſolid content of a ſaloon with a circular quadrantal arch of 2 feet radius, ſpringing over a rectangular room of 20 feet long and 16 feet wide?

Here the flat part of the cieling is 16 *feet by* 12; *and*

$$
\begin{array}{l}
4)56 \\
\overline{14} = \tfrac{1}{4} \text{ of the perimeter.} \\
\ \ 2 = \text{height.} \\
\overline{28} \\
\ \ 2 = \text{projection.} \\
\overline{55}
\end{array}
$$

ſimilar to that of the cieling; the ſide being equal to the difference of a ſide of the room, and a ſide of the cieling.

4. Add theſe three articles together, and the ſum will give the ſuperficial content required.

Note. In a rectangular, circular, or polygonal room, the baſe of the dome will be a ſquare, a circle, or a like polygon.

3.1416

```
   3.1416
       56
  -------
  188496
 157080
 --------
 175.9296=A.

 20=side of the room.
 16=side of the cieling.
 --
  4
  4
 --
 16
  1.000, &c.=factor.
 ------
 16.000
       1⅓=⅔ of the height.
 -------
 16.000
  5.333
 ------
 21.333=B.

  16
  12
 ---
 192=area of the flat cieling.
   2=height of the arch.
 ---
 384
 175.9296
  21.3333
 --------
 581.2629=solid content required.
```

2. A cir-

2. A circular building of 40 feet diameter, and 25 feet high to the cieling, is covered with a ſaloon, whoſe circular arch is 5 feet radius; required the capacity of the room in cubic feet?

Anſ. 30779.45948 *feet.*

PROBLEM VI.

To find the ſolid content of the vacuity formed by a groin arch, either circular or elliptical.

RULE.

Multiply the area of the baſe by the height, and the product again by .904, and it will give the ſolidity required.

EXAMPLES.

1. What is the ſolid content of the vacuity formed by a circular groin, one ſide of its ſquare baſe being 12 feet?

```
      12
      12
     ---
     144 = area of the baſe.
       6 = height.
     ---
     864
    .904
  ------
    3456
   77760
  ------
 781.056 = ſolidity required.
```

2. What

2. What is the ſolid content of the vacuity formed by an elliptical groin, one ſide of its ſquare baſe being 20 feet, and the height 6 feet? *Anſ.* 2169.6.

PROBLEM VII.

To find the concave ſuperficies of a circular groin.

RULE.*

Multiply the area of the baſe by 1.1416, and the product will be the ſuperficies required.

EXAMPLES.

1. What is the curve ſuperficies of a circular groin arch, one ſide of its ſquare baſe being 12 feet?

$$\begin{array}{r} 12 \\ 12 \\ \hline 144 = \textit{area of the base.} \\ 1.1416 \\ \hline 45664 \\ 45664 \\ 11416 \\ \hline 164.3904 = \textit{superficies required.} \end{array}$$

2. What

* This rule may alſo be obſerved in elliptical groins, the error being too ſmall to be regarded in practice.

In meaſuring works where there are many groins in a range, the cylindrical pieces between the groins, and on their ſides, muſt be computed ſeparately.

2. What is the concave ſuperficies of a circular groin arch, one ſide of its ſquare being 9 feet?

Anſ. 92.4696.

And to find the ſolidity of the brick, or ſtone work, which forms the groin arches, obſerve the following

Rule. Multiply the area of the baſe by the height, including the work over the top of the groin, and this product leſſened by the ſolid content, found as before, will give the ſolidity required.

The general rule for meaſuring all arches is this:

From the content of the whole, conſidered as ſolid, from the ſpringing of the arch to the outſide of it, deduct the vacuity contained between the ſaid ſpringing and the under ſide of it, and the remainder will be the content of the ſolid part.

And becauſe the upper ſides of all arches, whether vaults or groins, are built up ſolid, above the haunces, to the ſame height with the crown, it is evident that the area of the baſe will be the whole content above-mentioned, taking for its thickneſs the height from the ſpringing to the top. And for the content of the vacuity to be deducted, take the area of its baſe, accounting its thickneſs to be $\frac{2}{3}$ of the greateſt inſide height. But it may be noted that the area uſed in the vacuity, is not exactly the ſame with that uſed in the ſolid; for the diameter of the former is twice the thickneſs of the arch leſs than that of the latter.

And although I have mentioned the deduction of the vacuity as common to both the vault and the groin, it is reaſonable to make it only in the former, on account of the waſte of materials and trouble to the workman, in cutting and fitting them for the angles and interſections.

Whoever wiſhes to ſee this ſubject more fully handled, may conſult *La Théorie et la Pratique de la Géométrie*, par *M. l'Abbé Deidier*; a work in which ſeveral parts of Menſuration and Practical Geometry are ſkilfully handled, the examples being moſtly wrought out in an eaſy familiar manner, and illuſtrated with obſervations, and figures very neatly executed.

OF

OF THE

CARPENTER's RULE.

THIS instrument is commonly called *Cogeshall's* sliding rule. It consists of two pieces, of a foot in length each, which are connected together by means of a folding joint.

On one side of the rule, the whole length is divided into inches and half quarters, for the purpose of taking dimensions. And on this face there are also several plain scales, divided by diagonal lines into twelfth parts, which are designed for planning such dimensions as are taken in feet and inches.

On one part of the other face there is a slider, and four lines marked A, B, C, and D; the two middle ones B and C being upon the slider.

Three of these lines A, B, C, are double ones, because they proceed from one to 10 twice over; and the fourth line D is a single one, proceeding from 4 to 40, and is called the *girt line*.

The use of the double lines, A and B, is for working proportions, and finding the areas of plane figures. And the use of the girt line D, and the other double line C, is for measuring solids.

When 1 at the beginning of any line is counted 1, then the 1 in the middle will be 10, and the 10 at the end 100. And when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000, &c. and all the small divisions are altered in value accordingly.

Upon the other part of this face there is a table of the value of a load of timber, at all prices, from 6*d.* to 2*s.* a foot.

Some rules have likewiſe a line of inches, or a foot divided decimally into 10th parts; as well as tables of board meaſure, timber meaſure, &c. but theſe will be beſt underſtood from a ſight of the inſtrument.

The uſe of the SLIDING RULE.

PROBLEM I.

To find the product of two numbers, as 7 *and* 26.

RULE.

Set 1 upon A, to one of the numbers (26) upon B; then againſt the other number (7) on A, will be found the product (182) upon B.

Note. If the third term runs beyond the end of the line, ſeek it on the other radius, or part of the line, and increaſe the product 10 times.

PROBLEM II.

To divide one number by another, as 510 *by* 12.

RULE.

Set the diviſor (12) on A, to 1 on B; then againſt the dividend (510) on A, is the quotient ($42\frac{1}{2}$) on B.

Note. If the dividend runs beyond the end of the line, diminiſh it 10 or 100 times to make it fall on A, and increaſe the quotient accordingly.

PRO-

PROBLEM III.

To square any number, as 27.

RULE.

Set 1 upon D to 1 upon C; then against the number (27) upon D, will be found the square (729) upon C.

If you would square 270, reckon the 1 on D to be 100; and then the 1 on C will be 1000, and the product 72900.

PROBLEM IV.

To extract the square root of any number, as 4268.

RULE.

Set 1 upon C, to 1 upon D; then against (4268) the number on C, is (65.3) the root on D.

To value this right you must suppose the 1 on C to be some of these squares 1, 100, 1000, &c. which is the nearest to the given number, and then the root corresponding will be the value of the 1 upon D.

PROBLEM V.

To find a mean proportional between any two numbers, as 27 and 450.

RULE.

Set one of the numbers (27) on C, to the same on D; then against the other number (450) on C, will be the mean (112) on D.

Note. If one of the numbers overruns the line, take the 100th part of it, and augment the answer 10 times.

PROBLEM VI.

Three numbers being given to find a fourth proportional; suppose 12, 28, and 57.

RULE.*

Set the first number (12) upon A, to the second (28) upon B; then against the third number (57) on A, is the fourth (133) on B.

Note. If one of the middle numbers runs off the line, take the 10th part of it only, and augment the answer 10 times.

The finding a third proportional is exactly the same, the second number being twice repeated.

Thus, suppose a third proportional was required to 21 and 32.

Set the first 21 on B, to the second 32 on A; then against the second 32 on B, is 48.8 on A, which is the third proportional required.

---

* The use of the rule in board and timber measure will be shewn in what follows.

If the breadth of a board be given; to find how much in length will make a square foot.

Rule. If the board be narrow, it will be found in the table of board measure on the rule; but, if not, shut the rule, and seek the breadth in the line of board measure, running along the rule, from that table; then over against it, on the opposite side, is the length in inches required.

The side of the square of a piece of timber being given; to find how much in length will make a foot solid.

RULE. If the timber be small, it will be found in the table of timber measure on the rule; but, if not, look for the side of the square, in the line of timber measure, running along the rule, from that table, and against it in the line of inches is the length required.

OF

OF

TIMBER MEASURE.

PROBLEM I.

To find the area, or superficial content, of a board or plank.

RULE.

MULTIPLY the length by the breadth, and the product will be the content required.

Note. When the board is tapering, add the breadths of the two ends together, and take $\frac{1}{2}$ the sum for the mean breadth.

By the SLIDING RULE.

Set 12 on B to the breadth in inches on A, then against the length in feet on B, is the content on A, in feet and fractional parts, as required.

EXAMPLES.

1. What is the value of a plank, whose length is 8 feet 6 inches, and breadth 1 foot 3 inches throughout, at $2\frac{1}{2}d.$ per foot?

feet	*in.*		
8	6		
1	3		
8	6		
2	1	6	
10	7	6	*the content.*

	10 7 6
2*d.* is $\frac{1}{6}$	1 8
$\frac{1}{2}$ is $\frac{1}{4}$	5
in. 6 is $\frac{1}{2}$	$1\frac{1}{4}$
in. 1 is $\frac{1}{6}$	$\frac{1}{4}$
	2*s.* $2\frac{1}{2}$*d.* *the answer.*

By the SLIDING RULE.

As 12 *on B* : 15 *on A* :: $8\frac{1}{2}$ *on B* : $10\frac{1}{2}$ *on A.*

2. What is the content of a board, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches?

feet in. pa.
Ans. 10 2 10

3. At $1\frac{1}{2}$*d.* per foot, what is the value of a plank, whose length is 12 feet 6 inches, and breadth 11 inches throughout? *Ans.* 1*s.* 5*d.*

4. Find the value of 5 oaken planks at 3*d.* per foot, each being $17\frac{1}{2}$ feet long, and their particular breadths as follows: viz. two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower? *Ans.* 1*l.* 5*s.* $9\frac{1}{4}$*d.*

PROBLEM II.

To find the solidity of squared or four-sided timber.

RULE.

RULE.*

Multiply the mean breadth by the mean thickneſs, and this product again by the length, and it will give the ſolidity required.

By the Sliding Rule.

As the length in feet on *C* : 22 on *D* :: quarter girt in inches on *D* : ſolidity on *C*.

---

* *Note* 1. If the ſtick be equally broad and thick throughout, the breadth and thickneſs, any where taken, will be the mean breadth and thickneſs.

2. If the tree tapers regularly from one end to the other, the breadth and thickneſs, taken in the middle, will be the mean breadth and thickneſs.

3. If the ſtick does not taper regularly, but is thicker in ſome places than in others, let ſeveral different dimenſions be taken, and their ſum divided by the number of them will give the mean dimenſions.

This method of finding the mean dimenſions is moſtly uſed in practice, but, in many caſes, it is exceedingly erroneous.

The quarter girt, likewiſe, which is mentioned in the proportion by the ſliding rule, is ſubject to error. It is not the fourth part of the circumference, but the ſquare root of the product ariſing from multiplying the mean breadth by the mean thickneſs.

In order to ſhew the fallacy of taking $\frac{1}{4}$ of the girt for the ſide of a mean ſquare, take the following example:

Suppoſe a piece of timber to be 24 feet long, and a foot ſquare throughout, and let it be ſlit into two equal parts, from end to end.

Then the ſum of the ſolidities of the two parts, by the quarter girt method, will be 27 feet, but the true ſolidity is 24 feet, and if the two pieces were very unequal, the difference would be ſtill greater.

EXAMPLES.

1. The length of a piece of timber is 20½ feet, the breadth at the greater end is 1 foot 9 inches, and the thickneſs 1 foot 3 inches; and at the leſſer end the breadth is 1 foot 6 inches, and the thickneſs 1 foot: what is the ſolidity?

```
   1.75 = greater breadth.
   1.5  = leſſer breadth.
 2)3.25
   1.625 = mean breadth.

   1.25 = greater thickneſs.
   1.00 = leſſer thickneſs.
 2)2.25
   1.125 = mean thickneſs.
```

By Decimals.

```
      1.625
      1.125
      8125
     3250
    1625
   1625
   1.828125
       20.5
    9140625
   36562500
 37.4765625 = content.
```

By

By Cross Multiplication.

fo.	in.	pa.		
1	7	6		
1	1	6		
1	7	6		
	1	7	6	
		9	9	
1	9	11	3	
20	6			
36	6	9	0	
	10	11	7	6
37	5	8	7	6 = *content.*

By the SLIDING RULE.

As 1 *upon B* : $19\frac{6}{12}$ *upon A* :: $13\frac{6}{12}$ *upon B* : $263\frac{25}{100}$ *upon A, the mean square.*

As 16 *upon C* : 4 *upon D* :: 1.8 *upon C* : 16.2 *upon D, the side of the mean square.*

As $20\frac{1}{2}$ *upon C* : 12 *upon D* :: 16.2 *upon D* : $37\frac{5}{12}$ *upon C, the answer.*

2. The length of a piece of timber is 24.5 feet, and its ends are equal squares, whose sides are each 1.04 feet: what is the solidity? *Ans.* 26 *feet* 6 *inches.*

3. The length of a piece of timber is 20.38 feet, and the ends are unequal squares, the side of the greater being $19\frac{1}{8}$ inches, and that of the lesser $9\frac{1}{8}$ inches: what is the solidity? *Ans.* 29 *feet* 4 *inches.*

4. The length of a piece of timber is 27.36 feet; at the greater end the breadth is 1.78 feet, and the thickness 1.23 feet; and at the lesser end the breadth is 1.04 feet, and the thickness .91 feet: what is the solidity? *Ans.* 41.726 *feet.*

PROBLEM III.

To find the solidity of round or unsquared timber.

RULE I.*

Multiply the square of the quarter girt (or $\frac{1}{4}$ of the circumference) by the length, and the product will be the content, *according to the common practice.*

By the SLIDING RULE.

As the length upon C : 12 upon D : : $\frac{1}{4}$ girt upon D : content upon C.

EXAMPLES.

1. A piece of timber is $9\frac{3}{4}$ feet long, and the quarter girt is 39 inches: what is the solidity?

---

* Let c = girt or circumference, and l = length of the tree.

Then $\frac{c}{4} \times \frac{c}{4} \times l = \frac{c^2 l}{16}$ = content of the tree according to the rule.

And $\frac{c^2}{4 \times 3.1416} \times l = \frac{c^2 l}{12.5664}$ = true content, according to the rule for finding the content of a cylinder.

But $\frac{c^2 l}{12.5664}$ differs from $\frac{c^2 l}{16}$ by nearly $\frac{1}{4}$ part of the whole, and therefore the rule is exceedingly erroneous.

When the tree is tapering, the mean girt is found in the same manner as in board measure. Or if the tree be very irregular, the best way is to divide it into a certain number of lengths, and find the content of each part separately.

When trees have their bark on, an allowance is generally made, by deducting so much from the girt as is judged sufficient to reduce it to such a circumference as it would have without its bark. In oak this allowance is about $\frac{1}{10}$ or $\frac{1}{12}$ part of the girt; but for elm, beach, ash, &c. whose bark is not so thick, the deduction ought to be less.

By

By Decimals.

```
     3.25 = 39 inches.
     3.25
     ----
     1625
     650
    975
   -------
  10.5625
     9.75 = 9¾ feet.
  -------
   528125
  739375
 950625
 ----------
 102.984375 = solidity.
```

By Crofs Multiplication.

```
  f.  in.
  3   3 = 39 inches.
  3   3
  -----
  9   9
      9   9
 ----------
 10   6   9
  9   9     = 9¾ feet.
 ----------
 95   0   9
  7  11   0   9
 ---------------
102  11   9   9 = solidity
```

By the SLIDING RULE.

As 9¾ *upon* C : 12 *upon D* : : 39 *upon D* : 103 *upon* C, *the content*.

2. The

2. The length of a tree is 25 feet, and the girt throughout $2\frac{1}{2}$ feet: what is its ſolidity?

Anſ. 9 feet 9 inches.

3. The length of a tree is $14\frac{1}{2}$ feet, and its girt in the middle 3.15 feet: required the ſolidity?

Anſ. 9 feet nearly.

4. The girts of a tree in 4 different places are as follows: in the firſt place 5 feet 9 inches, in the ſecond 4 feet 5 inches, in the third 4 feet 9 inches, and in the fourth 3 feet 9 inches; and the length of the whole tree is 15 feet: what is the ſolidity?

Anſ. 20 feet 5 inches.

5. An oak tree is 45 feet 7 inches long, and its quarter girt 3 feet 8 inches; what is the ſolid content, allowing $\frac{1}{12}$ for the bark? *Anſ. 515 feet, nearly.*

RULE II.*

Multiply the ſquare of $\frac{1}{5}$ of the girt by twice the length, and the product will be the ſolidity, *extremely near the truth.*

By

---

* Let c = circumference, and l = length, as before.

Then $\frac{c}{5} \times \frac{c}{5} \times \frac{2l}{1} = \frac{2c^2l}{25} = \frac{c^2l}{12.5}$ = content of the tree according to the rule.

And the true content is $= \frac{c^2l}{12.5664}$, as was before ſhown.

But $\frac{c^2l}{12.5}$ differs from $\frac{c^2l}{12.5664}$ by only about $\frac{1}{190}$ part of the whole, and is therefore ſufficiently near the truth for any practical purpoſe.

This rule is full as eaſy in practice as the falſe one, and therefore ought to be generally uſed, ſince the eaſe of the other method is the only argument which is alledged for employing it.

5 The

By the SLIDING RULE.

At twice the length upon C : 12 upon D :: $\frac{1}{5}$ of the girt upon D : content upon C.

EXAM-

The following rule was given me by *Mr. Burrow*, and is a ſtill nearer approximation.

Rule. Multiply the ſquare of the circumference by the length, and take $\frac{1}{11}$ of the product; from this laſt number ſubtract $\frac{1}{8}$ of itſelf, and the remainder will be the anſwer.

For $\frac{c^2 l}{12.5664} = \frac{7}{88}$ of $c^2 l$ *very nearly*, $= \frac{8}{88} - \frac{1}{88} \times c^2 l = \frac{c^2 l}{11} - \frac{1}{8}$ of $\frac{c^2 l}{11}$, which is the ſame as the rule; and differs from the truth by only 1 foot in 2300.

The following problems, as well as many other things in this ſection, were taken from *Dr. Hutton*'s Menſuration. They are given to ſhew the artifices that may be uſed in meaſuring timber according to the falſe method now practiſed, and the abſolute neceſſity there is of aboliſhing it.

PROBLEM I.

To find where a tree muſt be cut, ſo that the two parts, meaſured ſeparately, ſhall produce a greater ſolidity than that of the whole tree, or any other two parts of it.

RULE. Cut it through exactly in the middle, or at $\frac{1}{2}$ of the length, and the two parts will meaſure the moſt poſſible.

Demon. Put G = greateſt girt, g = leaſt, x = girt at the ſection, L = whole length, and z = length to be cut off the leſs end.

Then, by ſimilar figures, L : z :: G $- g$: $x - g$; or $x = \frac{Gz - gz}{L} + g$.

But $\overline{g+x}|^2 \times z + \overline{G+x}|^2 \times \overline{L-z}$ = a maximum.

And if the fluxion of this expreſſion be put equal to nothing, and the value of x be ſubſtituted inſtead of x, there will reſult $z = \frac{1}{2}$L. Q. E. D.

Example.

EXAMPLES.

1. A piece of timber is $9\frac{3}{4}$ feet long, and $\frac{1}{5}$ of the girt is 2.6 feet: what is the solidity?

By

Example. Suppoſing a tree to girt 14 feet at the greater end, 2 feet at the leſs, and 8 feet in the middle, and that the length is 32 feet.

Then, by the common method, the whole tree meaſures only 128.

But, when cut through the middle, the greater part meaſures 121 feet, and the leſs part 25 feet.

And the ſum of theſe two parts is 146 feet, which exceeds the whole by 18 feet, and is the moſt that it can be made to meaſure by cutting it into two parts.

PROBLEM II.

To find where a tree muſt be cut, ſo that the part next the greater end may meaſure the moſt poſſible.

Rule. Cut it where the girt is $\frac{1}{3}$ of the greateſt girt, and the greater end will meaſure to the moſt poſſible.

Demon. By uſing the ſame notation as in the laſt problem, we ſhall have $x = \frac{Gz - gz}{L} + g$, and $\overline{G + x} \times \overline{L - z} =$ a maximum.

And, if this be put into fluxions, it will give $z = \frac{G - 3g}{G - g} \times \frac{1}{3} L$, and $x = \frac{G - g}{L} \times z + g = \frac{1}{3} G$. Q. E. D.

Example. Taking here the ſame example as before, we ſhall have $12 : 8 :: \frac{32}{3} : 7\frac{1}{9} =$ length to be cut off, $24\frac{8}{9} =$ length of the remaining part, and $4\frac{2}{3} =$ girt at the ſection.

But the content of the whole tree is only 128 feet, and the content of the greater part $135\frac{41}{81}$ feet, which exceeds 128 by $7\frac{41}{81}$, and is the greateſt poſſible.

Note.

By Decimals.

```
     2.6
     2.6
     ---
     156
     52
    ----
    6.76
    9.75
    ----
    3380
   4732
  6084
  ------
  65.9100
        2
  -------
 131.8200 = Content.
```

By

Note. If the greateſt girt does not exceed 3 times the leaſt, the tree cannot be cut as is required by the problem; for when the leaſt girt is exactly equal to $\frac{1}{3}$ of the greater, the tree already meaſures to the moſt poſſible.

PROBLEM III.

To cut a tree, ſo that the part next the greater end may meaſure exactly the ſame as the whole tree.

Rule 1. Call the ſum of the girts of the two ends s, and their difference d.

2. Multiply d by the ſum of d and $4s$, and from the ſquare root of the product take the difference between d and $2s$.

3. Then as $2d$ is to this remainder, ſo is the whole length of the tree, to the length to be cut off the ſmaller end.

Demon.

By the SLIDING RULE.

As 19.15 *upon* C : 12 *upon* D : : $31\frac{1}{5}$ *in. upon* D : 132, *the content upon* C.

2. If the length of a tree be 24 feet, and the girt throughout 8 feet: what is the content?

Anf. 123 *feet, nearly.*

3. If a tree girt 14 feet at the thicker end, and 2 feet at the fmaller end: required the folidity when the length is 24 feet? *Anf.* $11\frac{1}{2}$ *feet, nearly.*

4. A tree girts in five different places as follows: in the firft place 9.43 feet, in the fecond 7.92 feet, in the third 6.15 feet, in the fourth 4.74 feet, and in the fifth 3.16 feet; and the whole length is $17\frac{1}{2}$ feet: what is the folidity? *Anf.* 54.4065 *feet.*

Demon. Ufing ftill the fame notation, we fhall have $s^2 L = \overline{L - z} \times \overline{G + x}|^2$; and by fubftituting for x its value $\frac{dz}{L} + g$, it will be $z = \frac{L}{2d} \times \sqrt{\overline{\overline{4s + d} \times d} - 2s + d}$, and $x = \frac{1}{2} \sqrt{\overline{4s + d} \times d} - s$. Q. E. D.

The rule may be illuftrated by taking the fame example as in the laft problems.

Thus, fince $s = 16$, $d = 12$, and the length $L = 32$, $\frac{32}{24} \times \sqrt{\overline{76 \times 12} - 20} = \frac{16}{3} \sqrt{57} - 26\frac{2}{3} = 13.599118 =$ length to be cut off; and therefore the length of the remaining part is 18.400882.

Alfo, $\frac{1}{2} \times \sqrt{\overline{76 \times 12}} - 8 = 2\sqrt{57} - 8 = 7.099669 =$ girt at the fection. Whence the girt in the middle of the greater part is $\frac{14 + 7.099669}{2} = 10.549834$, whofe $\frac{1}{4}$ part is 2.637458, and confequently the content of this part is $\overline{2.637458}|^2 \times 18.400882 = 128$ the fame as the content of the whole tree.

OF

OF

SPECIFIC GRAVITY.

THE ſpecific gravities of bodies are their relative weights contained under the ſame given magnitude, as a cubic foot, a cubic inch, &c.

The ſpecific gravities of ſeveral ſorts of bodies are expreſſed by the numbers annexed to their names in the following table.

A table of the ſpecific gravities of bodies.

Fine gold	19640	Brick	2000
Standard gold	18888	Light earth	1984
Quickſilver	14000	Solid gunpowder	1745
Lead	11325	Sand	1520
Fine ſilver	11091	Pitch	1150
Standard ſilver	10535	Dry box wood	1030
Copper	9000	Sea water	1030
Gun metal	8784	Common water	1000
Caſt braſs	8000	Dry oak	925
Steel	7850	Gunpowder, ſhaken	922
Iron	7645	Dry aſh	800
Caſt iron	7425	Dry maple	755
Tin	7320	Dry elm	600
Marble	2700	Dry fir	550
Common ſtone	2520	Cork	240
Loom	2160	Air	$1\frac{1}{4}$

Note. As a cubic foot of water weighs juſt 1000 ounces Avoirdupois, the numbers in this table expreſs not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each, in Avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be readily known.

PRO-

PROBLEM I.

To find the magnitude of a body from its weight being given.

RULE.

As the tabular ſpecific gravity of the body, is to its weight in Avoirdupois ounces,

So is one cubic foot, or 1728 cubic inches, to its content in feet, or inches, reſpectively.

EXAMPLES.

1. Required the content of an irregular block of common ſtone which weighs 1 *cwt.* or 112 *lbs.*?

```
             112 lbs.
              16
             ----
             672
             112
            -----
2520 : :    1792  : :  1728
            1728
           ------
           14336
           3584
          12544
          1792
         -------
2520)3096576(1228⅘ cubic inches the anſ.
     252
     ---
      576
      504
      ---
       725
       504
       ---
       2217
       2016
       ----
       2016
```

2. How

2. How many cubic inches of gunpowder are there in one pound weight? *Anſ.* 30 *nearly.*

3. How many cubic feet are there in a ton weight of dry oak? *Anſ.* $38\frac{138}{185}$.

PROBLEM II.

To find the weight of a body from its magnitude being given.

RULE.

As one cubic foot, or 1728 cubic inches, is to the content of the body,

So is its tabular ſpecific gravity, to the weight of the body.

EXAMPLES.

1. Required the weight of a block of marble, whoſe length is 63 feet, and its breadth and thickneſs each 12 feet; theſe being the dimenſions of one of the ſtones in the walls of Balbec.

$$63 \times 12 = 756$$
$$756 \times 12 = 9072$$
$$1 : 9072 :: 2700$$
$$9072 \times 2700 = 6350400 + 18144 = 24494400$$

16 { 4	24494400 } *oz.*
4	6123600
112	1530900 *lbs.*
20	13668 *cwts.*
	683 *ton.*

Anſ. $683\frac{2}{5}$ *tons, which is equal to the burthen of an Eaſt India ſhip.*

2. What

2. What is the weight of a pint of gunpowder ale-measure? *Anf.* 19 *oz. nearly.*

3. What is the weight of a block of dry oak, which meafures ten feet in length, 3 feet in breadth, and $2\frac{1}{4}$ feet deep? *Anf.* $4335\frac{15}{16}$ *lb.*

PROBLEM III.

To find the fpecific gravity of a body.

RULE.

Cafe 1. When the body is heavier than water, weigh it both in water and out of water, and the difference will be the weight loft in the water.

Then, as the weight loft in water, is to the whole weight,

So is the fpecific gravity of water, to the fpecific gravity of the body.

EXAMPLE.

A piece of ftone weighed in air 10 pounds, but in water only $6\frac{3}{4}$ lb. Required its fpecific gravity?

$$\begin{array}{r} 10 \\ 6\frac{3}{4} \\ \hline 3\frac{1}{4} \end{array} : 10 :: 1000 :$$

$$13 : 40 :: 1000 :$$

$$\begin{array}{r} 1000 \\ 40 \\ \hline 13)40000(3077 \\ 39 \\ \hline 100 \\ 91 \\ \hline 90 \end{array}$$

Cafe

Case 2. When the body is lighter than water, so that it will not quite sink; affix to it another body heavier than water, so that the mass compounded of the two may sink together.

Weigh the heavier body and the compound mass separately both in water and out of it, and find how much each loses in water, by subtracting its weight in water from its weight in air.

Then as the difference of these remainders is to the weight of the light body in air,

So is the specific gravity of water to the specific gravity of the body.

EXAMPLE.

Suppose a piece of elm weighs in air 15lb. and that a piece of copper which weighs 18lb. in air, and 16lb. in water, is affixed to it, and that the compound weighs 6lb. in water. Required the specific gravity of the elm?

	18	*in air*	33	
	16	*in water*	6	
loss	2		27	*loss*
			2	
			25	

25 : 15 :: 1000 : 600 *Ans.*

PROBLEM IV.

To find the quantities of two ingredients in a given compound.

RULE.

RULE.

Take the differences of every pair of the three specific gravities, *viz.* of the compound and each ingredient, and multiply the difference of every two by the third.

Then as the greatest product is to the whole weight of the compound, so is each of the other products to the weights of the two ingredients.

EXAMPLE.

A composition of 112lb. being made of tin and copper, whose specific gravity is found to be 8784; Required the quantity of each ingredient? The specific gravity of tin being 7320, and of copper 9000.

0000	9000	8784
7320	8784	7320
1680	216	1464 *diff.*
8784	7320	9000
702720	4320	13176000
52704	648	
8784	1512	
14757120	1581120	

14757120 : 112 :: 13176000
112

26352000
13176
13176

14757120)1475712000(100

Ans. 100*lb. of copper* and 12*lb. of tin* } *in the composition.*

MIS-

MISCELLANEOUS QUESTIONS.

1. WHAT difference is there between a floor 48 feet long, and 30 feet broad, and two others each of half the dimenſions? *Anſ.* 720 *feet.*

3. From a mahogany plank 26 inches broad, a yard and a half is to be ſawed off; what diſtance from the end muſt the line be ſtruck? *Anſ.* 6.23 *feet.*

3. A joiſt is $8\frac{1}{2}$ inches deep, and $3\frac{1}{2}$ broad: what will be the dimenſions of a ſcantling juſt as big again as the joiſt, that is $4\frac{3}{4}$ inches broad?

Anſ. 12.52 *inches deep.*

4. A roof is 24 feet 8 inches by 14 feet 6 inches, and is to be covered with lead at 8*lbs.* to the foot: what will it come to at 18*s.* per cwt?

Anſ. 22*l.* 19*s.* $10\frac{1}{4}$*d.*

5. What is the ſide of that equilateral triangle, whoſe area coſt as much paving at 8*d.* per foot, as the palliſading the three ſides did at a guinea per yard?

Anſ. 72.746 *feet.*

6. The two ſides of an obtuſe-angled triangle are 20 and 40 poles: what muſt the length of the third ſide be that the triangle may contain juſt an acre?

Anſ. 58.876, *or* 23.099.

7. If two ſides of a triangle, whoſe area is $60\sqrt{3}$, be 12 and 20: what is the third ſide? *Anſ.* 28.

8. If an area of 24 be cut off from a triangle, whoſe three ſides are 13, 14, and 15, by a line parallel to the longeſt ſide: what are the lengths of the ſides including that area?

Anſ. $\frac{13}{7}\sqrt{14}$, $2\sqrt{14}$, *and* $\frac{15}{7}\sqrt{14}$.

9. The diſtance of the centres of two circles, whoſe diameters are each 50, is equal to 30: what is the area of the ſpace incloſed by their circumferences?
Anſ. 559.119.

10. The area of an equilateral triangle, whoſe baſe falls on the diameter, and its vertex in the middle of the arc of a ſemi-circle, is equal to 100: what is the diameter of the ſemi-circle?
Anſ. 26.3214.

11. The four ſides of a field, whoſe diagonals are equal to each other, are 25, 35, 31, and 19 poles, reſpectively: what is the area?
Anſ. 4 *ac.* 1 *ro.* 38 *poles.*

12. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whoſe diameter is 289?
Anſ. 278.6716.

13. A cable which is 3 feet long, and 9 inches in compaſs, weighs 22*lbs.* what will a fathom of that cable weigh whoſe diameter is 9 inches?
Anſ. 434.26*lbs.*

14. A circular fiſh-pond is to be dug in a garden, that ſhall take up juſt half an acre, what muſt the length of the chord be that ſtrikes the circle?
Anſ. 27.75 *yards.*

15. A carpenter is to put an oaken curb to a round well, at 8*d.* per foot ſquare; the breadth of the curb is to be 7$\frac{1}{4}$ inches, and the diameter within 3$\frac{1}{2}$ feet: what will be the expence? *Anſ.* 5*s.* 2$\frac{1}{4}$*d.*

16. Suppoſe the expence of paving a ſemi circular plot, at 2*s.* 4*d.* per foot, amounted to 10*l.* what is the diameter of it? *Anſ.* 14.7737.

* 17. Seven men bought a grinding-ſtone of 60 inches in diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter muſt each grind

* For an excellent geometrical conſtruction of this queſtion, ſee *Dr. Hutton*'s Diarian Miſcellany, page 54.

down

down for his ſhare? *Anſ. The 1ſt.* 4.4508, *2d.* 4.8400, *3d.* 5.3535, *4th.* 6.0765, *5th.* 7.2079, *6th.* 9.3935, *and the 7th.* 22.6778.

18. A gentleman has a garden 100 feet long, and 80 feet broad, and a gravel walk is to be made of an equal width half round it: what muſt the width of the walk be ſo as to take up juſt half the ground?

Anſ. 25.968 *feet.*

19. In the midſt of a meadow well ſtored with graſs,
I took juſt an acre to tether my aſs;
How long muſt the cord be, that feeding all round,
He may'nt graze leſs or more than an acre of ground? *Anſ.* 39.25073 *yards.*

20. A maltſter has a kiln that is 16 feet 6 inches ſquare; now he wants to pull it down, and build a new one that will dry three times as much at a time as the old one did: what muſt be the length of its ſide?

Anſ. 28 *feet* 7 *inches.*

21. If a round ciſtern be 26.3 inches diameter, and 52.5 inches deep: how many inches diameter muſt a ciſtern be to hold twice the quantity, the depth being the ſame? *Anſ.* 37.19 *inches.*

22. A May-pole, whoſe top was broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the top of the pole: what was the height of the whole May-pole, ſuppoſing the length of the broken piece to be 39 feet? *Anſ.* 75 *feet.*

23. What will the diameter of a globe be, when the ſolidity and ſuperficial content thereof are equal to each other? *Anſ.* 6.

24. How many three inch cubes can be cut out of a 12 inch cube? *Anſ.* 64.

25. A farmer borrowed part of a hay-rick of his neighbour, which meaſured 6 feet every way, and paid him back again by two equal cubical pieces,

 each

each of whose sides were three feet: Query, whether the lender was fully paid? *Ans. He was paid ½ part only.*

26. What will the painting a conical church-spire come to at 8*d.* per yard; supposing the circumference of the base to be 64 feet, and the altitude 118 feet? *Ans.* 14*l.* 0*s.* 8¾*d.*

27. What will a marble frustum of a cone come to at 12*s.* per solid foot; the diameter of the greater end being 4 feet, that of the lesser end 1½ feet, and the length of the slant side 8 feet? *Ans.* 30*l.* 1*s.* 10*d.*

28. The diameter of a legal Winchester bushel is 18½ inches, and its depth 8 inches: what must the diameter of that bushel be whose depth is 7½ inches? *Ans.* 19.1071.

29. Three men bought a tapering piece of timber, which was the frustum of a square pyramid; one side of the greater end was 3 feet, one side of the lesser end 1 foot, and the length 18 feet: what is the length of each man's piece, supposing they paid equally, and are to have equal shares? *Ans.* 1*st.* 3.269, 2*d.* 4.559, *and the* 3*d.* 10.172, *reckoning from the greater end to the less.*

30. Suppose the ball at the top of St. Paul's Church is 6 feet in diameter: what did the gilding of it come to at 3½*d.* per square inch? *Ans.* 237*l.* 1*s.* 10¼*d.*

31. A person wants a cylindric vessel of 3 feet deep, that shall hold twice as much as a vessel of 28 inches deep, and 46 inches in diameter: what must be the diameter of the vessel required? *Ans.* 57.37 *inches.*

32. Two porters agreed to drink off a quart of strong beer between them, at two pulls, or a draught each; now, the first having given it a black eye, as it is called, or drank till the surface of the liquor touched the opposite edge of the bottom, gave the remaining

remaining part of it to the other: what was the difference of their ſhares, ſuppoſing the pot was the fruſtum of a cone, the depth being 5.7 inches, the diameter at the top 3.7 inches, and that of the bottom 4.23 inches? *Anſ.* 7.07 *cubic inches.*

33. Three perſons having bought a ſugar-loaf, want to divide it equally amongſt them by ſections parallel to the baſe; it is required to find the altitude of each perſon's ſhare, ſuppoſing the loaf to be a cone, whoſe height is 20 inches? *Anſ.* 13.867 *the upper part,* 3.604 *the middle part, and* 2.528 *the lower part.*

34. How high above the ſurface of the earth muſt a perſon be raiſed to ſee $\frac{1}{3}$ of its ſurface?

Anſ. To the height of the earth's diameter.

35. A cubical foot of braſs is to be drawn into a wire of $\frac{1}{40}$ of an inch in diameter; what will be the length of the wire, allowing no leſs in the metal?

Anſ. 97784.797 *yards, or near* 56 *miles.*

35. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raiſe one foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth muſt the ditch be dug, ſuppoſing its breadth to be every where 8 feet? *Anſ.* $7\frac{13}{168}$ *feet.*

36. Of what diameter muſt the bore of a cannon be, which is caſt for a ball of 24lbs. weight, ſo that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? *Anſ.* 5.757 *inches.*

37. At what height from the bottom muſt an upright cone be cut, ſo that the greateſt cylinder poſſible may be formed from the lower part of it?

Anſ. At $\frac{1}{3}$ *of the height.*

38. The ellipſe in Groſvenor ſquare, meaſures 840 links acroſs the longeſt way; and 612 the ſhorteſt, within the rails: now the walls being 14 inches thick, it is required to find what ground they incloſe, and

what they ſtand upon? *Anſ. They incloſe 4 ac.* 0 *ro.* 6 *po. and ſtand on* 1760$\frac{1}{2}$ *ſquare feet.*

49. If a heavy ſphere whoſe diameter is 4 inches, be put into a conical glaſs, full of water, whoſe diameter is 5, and altitude 6 inches; it is required to find how much water will run over?

Anſ. $\frac{35}{47}$ *of a pint nearly.*

40. Suppoſe it be found, by meaſurement, that a man of war, with its ordnance, rigging and appointments, draws ſo much water as to diſplace 50000 cubic feet of water: required the weight of the veſſel:

Anſ. 1395$\frac{1}{16}$ *tons.*

41. One ev'ning I chanc'd with a Tinker to ſit,
Whoſe tongue ran a great deal too faſt for his wit;
He talk'd of his art with abundance of mettle;
So I aſk'd him to make me a flat-bottom'd kettle:
Let the top and the bottom diameters be,
In juſt ſuch proportion as five is to three:
Twelve inches the depth I propos'd, and no more;
And to hold in ale gallons ſeven leſs than a ſcore.
He promis'd to do it, and ſtrait to work went;
But when he had done it he found it too ſcant.
He alter'd it then, but too big he had made it;
For though it held right, the diameters fail'd it:
Thus making it often too big and too little,
The Tinker at laſt had quite ſpoilt his kettle;
But he ſwears he will bring his ſaid promiſe to paſs,
Or elſe that he'll ſpoil every ounce of his braſs.
Now, to keep him from ruin, I pray find him out
The diameter's length, for he'll ne'er do it I doubt.

Anſ. The bottom diameter is 14.64017, *and the top diameter* 24.40028.

A

TABLE

OF THE

AREAS OF THE SEGMENTS OF A CIRCLE,

Whose Diameter is Unity, and supposed to be divided into 1000 equal Parts.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.001	.000042	.024	.004921	.047	.013392
.002	.000119	.025	.005230	.048	.013818
.003	.000219	.026	.005546	.049	.014247
.004	.000337	.027	.005867	.050	.014681
.005	.000470	.028	.006194	.051	.015119
.006	.000618	.029	.006527	.052	.015561
.007	.000779	.030	.006865	.053	.016007
.008	.000951	.031	.007209	.054	.016457
.009	.001135	.032	.007558	.055	.016911
.010	.001329	.033	.007913	.056	.017369
.011	.001533	.034	.008273	.057	.017831
.012	.001746	.035	.008638	.058	.018296
.013	.001968	.036	.009008	.059	.018766
.014	.002199	.037	.009383	.060	.019239
.015	.002438	.038	.009763	.061	.019716
.016	.002685	.039	.010148	.062	.020196
.017	.002940	.040	.010537	.063	.020680
.018	.003202	.041	.010931	.064	.021168
.019	.003471	.042	.011330	.065	.021659
.020	.003748	.043	.011734	.066	.022154
.021	.004031	.044	.012142	.067	.022652
.022	.004322	.045	.012554	.068	.023154
.023	.004618	.046	.012971	.069	.023659

The

The Areas of the Segments of a Circle.

Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.
.070	.024168	.103	.042687	.136	.064074
.071	.024680	.104	.043296	.137	.064760
.072	.025195	.105	.043908	.138	.065449
.073	.025714	.106	.044522	.139	.066140
.074	.026236	.107	.045139	.140	.066833
.075	.026761	.108	.045759	.141	.067528
.076	.027289	.109	.046381	.142	.068225
.077	.027821	.110	.047005	.143	.068924
.078	.028356	.111	.047632	.144	.069625
.079	.028894	.112	.048262	.145	.070328
.080	.029435	.113	.048894	.146	.071033
.081	.029979	.114	.049528	.147	.071741
.082	.030526	.115	.050165	.148	.072450
.083	.031076	.116	.050804	.149	.073161
.084	.031629	.117	.051446	.150	.073874
.085	.032186	.118	.052090	.151	.074589
.086	.032745	.119	.052736	.152	.075306
.087	.033307	.120	.053385	.153	.076026
.088	.033872	.121	.054036	.154	.076747
.089	.034441	.122	.054689	.155	.077469
.090	.035011	.123	.055345	.156	.078194
.091	.035585	.124	.056003	.157	.078921
.092	.036162	.125	.056663	.158	.079649
.093	.036741	.126	.057326	.159	.080380
.094	.037323	.127	.057991	.160	.081112
.095	.037909	.128	.058658	.161	.081846
.096	.038496	.129	.059327	.162	.082582
.097	.039087	.130	.059999	.163	.083320
.098	.039680	.131	.060672	.164	.084059
.099	.040276	.132	.061348	.165	.084801
.100	.040875	.133	.062026	.166	.085544
.101	.041476	.134	.062707	.167	.086289
.102	.042080	.135	.063389	.168	.087036

The

The Areas of the Segments of a Circle.

Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.
.169	.087785	.202	.113426	.235	.140688
.170	.088535	.203	.114230	.236	.141537
.171	.089287	.204	.115035	.237	.142387
.172	.090041	.205	.115842	.238	.143238
.173	.090797	.206	.116650	.239	.144091
.174	.091554	.207	.117460	.240	.144944
.175	.092313	.208	.118271	.241	.145799
.176	.093074	.209	.119083	.242	.146655
.177	.093836	.210	.119897	.243	.147512
.178	.094601	.211	.120712	.244	.148371
.179	.095366	.212	.121529	.245	.149230
.180	.096134	.213	.122347	.246	.150091
.181	.096903	.214	.123167	.247	.150953
.182	.097674	.215	.123988	.248	.151816
.183	.098447	.216	.124810	.249	.152680
.184	.099221	.217	.125634	.250	.153546
.185	.099997	.218	.126459	.251	.154412
.186	.100774	.219	.127285	.252	.155280
.187	.101553	.220	.128113	.253	.156149
.188	.102334	.221	.128942	.254	.157019
.189	.103116	.222	.129773	.255	.157890
.190	.103900	.223	.130605	.256	.158762
.191	.104685	.224	.131438	.257	.159636
.192	.105472	.225	.132272	.258	.160510
.193	.106261	.226	.133108	.259	.161386
.194	.107051	.227	.133945	.260	.162263
.195	.107842	.228	.134784	.261	.163140
.196	.108636	.229	.135624	.262	.164019
.197	.109430	.230	.136465	.263	.164899
.198	.110226	.231	.137307	.264	.165780
.199	.111024	.232	.138150	.265	.166663
.200	.111823	.233	.138995	.266	.167546
.201	.112624	.234	.139841	.267	.168430

The Areas of the Segments of a Circle.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.268	.169315	.301	.199085	.334	.229801
.269	.170202	.302	.200003	.335	.230745
.270	.171089	.303	.200922	.336	.231689
.271	.171978	.304	.201841	.337	.232634
.272	.172867	.305	.202761	.338	.233580
.273	.173758	.306	.203683	.339	.234526
.274	.174649	.307	.204605	.340	.235473
.275	.175542	.308	.205527	.341	.236421
.276	.176435	.309	.206451	.342	.237369
.277	.177330	.310	.207376	.343	.238318
.278	.178225	.311	.208301	.344	.239268
.279	.179122	.312	.209227	.345	.240218
.280	.180019	.313	.210154	.346	.241169
.281	.180918	.314	.211082	.347	.242121
.282	.181817	.315	.212011	.348	.243074
.283	.182718	.316	.212940	.349	.244026
.284	.183619	.317	.213871	.350	.244980
.285	.184521	.318	.214802	.351	.245934
.286	.185425	.319	.215733	.352	.246889
.287	.186329	.320	.216666	.353	.247845
.288	.187234	.321	.217599	.354	.248801
.289	.188140	.322	.218533	.355	.249757
.290	.189047	.323	.219468	.356	.250715
.291	.189955	.324	.220404	.357	.251673
.292	.190864	.325	.221340	.358	.252631
.293	.191775	.326	.222277	.359	.253590
.294	.192684	.327	.223215	.360	.254550
.295	.193596	.328	.224154	.361	.255510
.296	.194509	.329	.225093	.362	.256471
.297	.195422	.330	.226033	.363	.257433
.298	.196337	.331	.226974	.364	.258395
.299	.197252	.332	.227915	.365	.259357
.300	.198168	.333	.228858	.366	.260320

The

The Areas of the Segments of a Circle.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.367	.261284	.400	.293369	.433	.325900
.368	.262248	.401	.294349	.434	.326892
.369	.263213	.402	.295330	.435	.327882
.370	.264178	.403	.296311	.436	.328874
.371	.265144	.404	.297292	.437	.329866
.372	.266111	.405	.298273	.438	.330858
.373	.267078	.406	.299255	.439	.331850
.374	.268045	.407	.300238	.440	.332843
.375	.269013	.408	.301220	.441	.333836
.376	.269982	.409	.302203	.442	.334829
.377	.270951	.410	.303187	.443	.335822
.378	.271920	.411	.304171	.444	.336816
.379	.272890	.412	.305155	.445	.337810
.380	.273861	.413	.306140	.446	.338804
.381	.274832	.414	.307125	.447	.339798
.382	.275803	.415	.308110	.448	.340793
.383	.276775	.416	.309095	.449	.341787
.384	.277748	.417	.310081	.450	.342782
.385	.278721	.418	.311068	.451	.343777
.386	.279694	.419	.312054	.452	.344772
.387	.280668	.420	.313041	.453	.345768
.388	.281642	.421	.314029	.454	.346764
.389	.282617	.422	.315016	.455	.347759
.390	.283592	.423	.316004	.456	.348755
.391	.284568	.424	.316992	.457	.349752
.392	.285544	.425	.317981	.458	.350748
.393	.286521	.426	.318970	.459	.351745
.394	.287498	.427	.319959	.460	.352741
.395	.288476	.428	.320948	.461	.353739
.396	.289453	.429	.321938	.462	.354736
.397	.290432	.430	.322928	.463	.355732
.398	.291411	.431	.323918	.464	.356730
.399	.292390	.432	.324909	.465	.357727

The Areas of the Segments of a Circle.

Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.
.466	.358725	.478	.370706	.490	.382699
.467	.359723	.479	.371705	.491	.383699
.468	.360721	.480	.372704	.492	.384699
.469	.361719	.481	.373703	.493	.385699
.470	.362717	.482	.374702	.494	.386699
.471	.363715	.483	.375702	.495	.387699
.472	.364713	.484	.376702	.496	.388699
.473	.365712	.485	.377701	.497	.389699
.474	.366710	.486	.378701	.498	.390699
.475	.367709	.487	.379700	.499	.391699
.476	.368708	.488	.380700	.500	.392699
.477	.369707	.489	.381699		

THE END.

Published by the SAME AUTHOR,

1. THE SCHOLAR'S GUIDE TO ARITHMETIC. 5th Edit. Price 2s. bound.

2. AN INTRODUCTION TO ALGEBRA. 3d Edit.

3. AN INTRODUCTION TO ASTRONOMY, with many Plates. 2d Edit. Price 8s. bound.

4. ELEMENTS OF GEOMETRY. Containing the principal Propositions in the First Six and the Eleventh and Twelfth Books of Euclid, with Notes Critical and Explanatory. Price 6s. bound.